Mathematics X

Based on the syllabus prescribed by the BSEM

For Class X

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Cover Picture: Fractals are any of various extremely irregular curves or shapes for which any suitably chosen part is similar in shape to a given larger or smaller part when magnified or reduced to the same size.

Front Cover: (a) $2^{82589933} - 1$ is the largest known prime (as of January 2019) with 24862048 digits. It is a Mersenne prime (a prime of the form $2^p - 1$, where p is a prime). (b) The diagram shows the construction of a pair of tangents to a circle from an external point.

Back Cover: RSA-768 is a semiprime (a number with exactly two prime factors). It was a part of the RSA Factoring Challenge. Large semiprimes play an important role in cryptography.

Mathematics X

Price: ₹ 400/-

First Edition 2015
Second Edition 2019
This edition is published by Lousing Chaphu, Thoubal.
info[at]lousingchaphu.com
https://www.lousingchaphu.com

Printed in Manipur by International Printers, Thoubal Melaground, Thoubal 795138.

To teachers who inspire the young minds of tomorrow

Preface

This edition is equipped with more than a hundred pages of new materials. Unlike the first edition, this edition includes complete proofs of almost all theorems. We have added numerous examples and exercises. The purpose of these examples and exercises is to develop problem solving skills. It is through these examples and exercises that the students can assess progress and understanding. Hints are given to several of the exercises.

Our general approach on this book is simple, and we hope that the students will be equally interested in all parts of this book. The main emphasis is given on understanding the underlying fundamental principles. This book can be used as a main text, a reference text or a supplementary text. It can also be used for self study.

This new edition gave us the opportunity to streamline some arguments to correct errors and misprints and to ratify proofs and solutions. We would like to thank all those who gave comments and suggestions to the first edition. We welcome remarks and suggestions from our readers.

Thoubal
February 2019

A J Sanasam
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Acknowledgements

This work has been made possible with the support and encouragement from several persons. First of all, we are grateful to Hidam Gourashyam for his encouragement and mental support.

The publication of this book would have been delayed if it were not for the support given by Laishram Aboy Singh, Laishram Wanglen Meitei, Phaln Maibam, Rabinath Akham, Md Gayaz Ahamed, Puyam Ajit Singh and Mayanglambam Stalin Singh.

Finally, we would like to thank Jasoba Singh Konthoujam, Arjun Laishram, Oinam Premdas Singh and Jocyline Thokchom for their endless support.

Contents

Notations and Abbreviations

$\mathbb{N}$	the natural numbers
$\mathbb{Z}$	the integers
$\mathbb{Q}$	the rational numbers
$\mathbb{R}$	the real numbers
$\mathbb{C}$	the complex numbers

$\Longleftrightarrow$	if and only if
$\Longrightarrow$	implies
$\forall$	for all
$\exists$	there exists
$\triangle$	triangle
$\angle$	angle
$x \in A$	the element x belongs to the set A
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
$n!$	the product of first n natural numbers
$\lfloor x \rfloor$	the greatest integer less than or equal to x
$\lvert x \rvert$	the absolute value of x
(a, b)	the gcd of a and b (see Chapter 1)
$[a, b]$	the lcm of a and b
(x, y)	the coordinates of a point (see Chapter 11)
$\min\{a, b\}$	the minimum of a and b
$\max\{a, b\}$	the maximum of a and b
$\square$	Q.E.D. (quod erat demonstrandum), that which was to be demonstrated

CMO	Canadian Mathematical Olympiad
IMO	International Mathematical Olympiad
INMO	Indian National Mathematical Olympiad
USAMO	United States of America Mathematical Olympiad

Chapter 1

Number System

"Begin at the beginning," the King said gravely, "and go on till you come to the end: then stop."

— Lewis Carroll, *Alice in Wonderland*

A number is an abstract (mathematical) object used to count, measure and label. A number system is a writing system to represent numbers. The set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers, etc., are some commonly used number systems.

Divisibility

Notation: In this chapter, the small letters a, b, c, d, k, n, etc., denote integers unless stated otherwise; they can be positive, negative, or zero.

Definition 1.1 (Divisibility). An integer a is said to divide an integer b if there exists an integer c such that $b = ac$. In this case, we say *a divides b* or *b is divisible by a* or *a is a divisor* (or *factor*) *of b* or *b is a multiple of a* and we write $a \mid b$. We write $a \nmid b$ if a does not divide b. The notation $a \mid b$ should not be confused with the fraction a/b.

***Do you know*?** Zero divides only zero. If $0 \mid n$, then $n = 0 \times a$ for some integer a, i.e., $n = 0$ and hence $0 \mid 0$. However, the division of zero by zero $(0/0)$ is indeterminate.

Definition 1.2 (Prime number). An integer is prime if it is greater than 1 and it has exactly two distinct positive factors viz. 1 and the number itself. Equivalently, an integer is prime if it is greater than 1 and whenever it divides a product, it divides at least one of the factors.

Definition 1.3 (**Composite number**)**.** An integer is composite if it has more than two distinct positive factors.

Remark: 1 is neither prime nor composite. 2 is the only even prime.

Definition 1.4 (**HCF** or **GCD**)**.** The *highest common factor* or *greatest common divisor* of two integers a and b is the integer d satisfying the following properties:

(i) d is non-negative, $(d \geq 0)$,

(ii) d divides both a and b, $(d \mid a$ and $d \mid b)$,

(iii) every common divisor of a and b divides d, $(e \mid a,\ e \mid b \implies e \mid d)$.

The HCF of a and b is denoted by $\gcd(a, b)$ or simply by (a, b). In particular, $(a, b) = 0$ if and only if $a = b = 0$. Otherwise $(a, b) \geq 1$.

Remark: Most authors define HCF for two integers not both zero.

Definition 1.5 (**Relatively prime** or **coprime**)**.** Two integers a and b are said to be relatively prime (or coprime, or prime to each other) if $(a, b) = 1$.

In other words, a and b are relatively prime if and only if their only common divisors are ± 1.

Definition 1.6 (**LCM**)**.** The *least common multiple* of two integers a and b is the integer d satisfying the following properties:

(i) d is non-negative, $(d \geq 0)$,

(ii) d is divisible by both a and b, $(a \mid d$ and $b \mid d)$,

(iii) d divides every common multiple of a and b, $(a \mid e,\ b \mid e \implies d \mid e)$.

The LCM of a and b is denoted by $\operatorname{lcm}[a, b]$ or simply by $[a, b]$.

Theorem 1.1. *The product of two positive integers is equal to the product of their HCF and LCM. In general,* $(a, b)[a, b] = |ab|$ *for any* $a, b \in \mathbb{Z}$*.*

Note: The product of three (or more) positive integers need not be equal to the product of their HCF and LCM. However, the following results hold good for *three* positive integers a, b, c:

$$(a, b, c) = \frac{abc[a, b, c]}{[a, b][b, c][c, a]} \quad \text{and} \quad [a, b, c] = \frac{abc(a, b, c)}{(a, b)(b, c)(c, a)}.$$

Lemma: A lemma is a provable statement used in proving another statement.
Algorithm: An algorithm is a well defined sequence of steps forming a process for solving a given problem.

Euclid's Division Lemma

Theorem 1.2 (Euclid's division lemma). *Let a and b be any two integers and $b > 0$. Then there exist unique integers q and r such that $a = bq + r$ and $0 \leq r < b$.*

The integer q is called the quotient of a with respect to b and the integer r is called the remainder of a with respect to b.

Example 1. Show that every integer is of the form $2k$ or $2k + 1$.

Solution: Let a be any integer. Applying Euclid's division lemma on a and 2, we have

$$a = 2k + r, \quad 0 \leq r < 2, \quad \text{where } k \text{ and } r \text{ are integers.}$$

Here, $r = 0$ or 1. Hence, every integer is of the form $2k$ or $2k + 1$. □

Exercise 2. Show that every integer is of the form $3k$, $3k + 1$ or $3k + 2$.

Exercise 3. Is it true that the values of the remainder when a positive integer is divided by 4 are 0 and 1 only? Justify your answer.

Exercise 4. The square of any positive integer cannot be of the form $3m + 2$, where m is a natural number. Justify.

Exercise 5. Show that every odd integer is of the form $4k + 1$ or $4k + 3$.

Exercise 6. Show that every square integer is of the form $4k$ or $4k + 1$.

Exercise 7. Prove that one of every three consecutive integers is divisible by 3.

Example 8. Show that the sum of the squares of two odd integers is of the form $4k + 2$.

Solution: Let m and n be any two odd integers. We know that every odd integer is of the form $2q + 1$, for some integer q. So, $m = 2a + 1$ and $n = 2b + 1$ for some integers a and b.

$$\begin{aligned}
\therefore\ m^2 + n^2 &= (2a+1)^2 + (2b+1)^2 \\
&= 4a^2 + 4a + 1 + 4b^2 + 4b + 1 \\
&= 4(a^2 + a + b^2 + b) + 2 \\
&= 4k + 2, \quad \text{where } k = a^2 + a + b^2 + b \in \mathbb{Z}.
\end{aligned}$$

This shows that the sum of the squares of two odd integers is of the form $4k + 2$. □

Exercise 9. If a is an odd integer, show that $a^2 + (a+2)^2 + (a+4)^2 + 1$ is divisible by 12.

Hint: Take $a = 2k + 1$.

Exercise 10. Prove that the sum of two consecutive odd numbers is divisible by 4.

Exercise 11. Let m and n be any two odd integers. Prove that m^2+n^2+6 is divisible by 8.

Exercise 12. If $p \geq 5$ is a prime, show that $p^2 + 2$ is divisible by 3.

Hint: p is of the form $3k + 1$ or $3k + 2$.

Exercise 13. Show that every positive odd integer is of the form $6q + 1$, $6q + 3$ or $6q + 5$, where q is some integer.

Exercise 14. Show that the cube of any positive integer is of the form $9m$, $9m + 1$ or $9m + 8$.

Hint: Every positive integer is of the form $3k$, $3k + 1$ or $3k + 2$.

Exercise 15. If an integer n is a square as well as a cube (like $64 = 8^2 = 4^3$), then prove that n must be of the form $7k$ or $7k + 1$.

Hint: Every square is of the form $7k$, $7k + 1$, $7k + 2$ or $7k + 4$, and every cube is of the form $7k$, $7k + 1$ or $7k + 6$.

Exercise 16. Prove that every prime number greater than 3 is of the form $6k + 1$ or $6k + 5$, where k is some integer.

Hint: $6k + r$ is divisible by 2 if $r = 0, 2, 4$ and is divisible by 3 if $r = 3$.

Exercise 17. Let r be the remainder obtained by dividing a prime number p by 30. Show that either $r = 1$ or r itself is a prime number.

Exercise 18. Prove that no integer in the sequence $11, 111, 1111, \ldots$ is a perfect square.

Hint: Every number in the sequence is of the form $4k + 3$.

Exercise 19. If $n \geq 4$, then prove that $n, n+2, n+4$ cannot all be primes.

Hint: Every integer is of the form $3k$, $3k + 1$ or $3k + 2$.

Exercise 20. If n is an odd integer, prove that $n^4 + 4n^2 + 11$ is divisible by 16.

Exercise 21. If p and $8p-1$ are both prime, prove that $8p+1$ is composite.

Hint: If $p = 3k + 2$, then $8p - 1$ is divisible by 3. $p = 3$ or $p = 3k + 1$.

Euclid's Algorithm

Euclid's algorithm for finding the HCF of two given positive integers:

Step 1. Find the quotient and remainder of the division of the greater number by the smaller.

Step 2. If the remainder is zero, then the divisor is the HCF.

Step 3. Else, taking the previous remainder as the new divisor and the previous divisor as the new dividend, find the quotient and the remainder.

Step 4. Continue the process till the remainder is zero. The last divisor is the required HCF.

Remark: Although Euclids algorithm is stated for only positive integers, it can be extended for all non-zero integers. Euclid's algorithm works because of the following result.

Theorem 1.3. *If $a = bq + r$, then $(a, b) = (b, r)$.*

Proof: If $(a, b) = d$, then $d \mid r$. If $c \mid b$ and $c \mid r$, then $c \mid a$. Now, $c \mid a,\ c \mid b \implies c \mid d$. Thus, d divides both b and r, and any common divisor c of b and r also divides d. Hence, $(b, r) = d$.

Example 22. Let a and b be two positive integers where $a > b$. Prove that the last divisor (or the last non-zero remainder) in the Euclid's algorithm for a and b is the HCF of a and b.

Solution: Applying Euclid's division lemma successively, we get the following series of relations:

$$\begin{aligned}
a &= bq + r_1, & 0 < r_1 < b, && (1)\\
b &= r_1q_1 + r_2, & 0 < r_2 < r_1, && (2)\\
r_1 &= r_2q_2 + r_3, & 0 < r_3 < r_2, && (3)\\
\vdots\ & \qquad \vdots & \vdots\qquad && \\
r_{n-2} &= r_{n-1}q_{n-1} + r_n, & 0 < r_n < r_{n-1}, && (n)\\
r_{n-1} &= r_nq_n + 0. & && (n+1)
\end{aligned}$$

Here, $q, q_1, \dots, q_n$, $r_1, r_2, \dots, r_n$ are positive integers and $a > b > r_1 > r_2 > \dots > r_n$. The set of equations (1) to $(n+1)$ is the Euclid's algorithm for a and b.

We know that if $a = bq + r$, then $(a,b) = (b,r)$. Applying this result successively to the equations (1) to $(n+1)$, we get

$$(a,b) = (b,r_1) = (r_1,r_2) = \cdots = (r_{n-1},r_n) = r_n.$$

The last equality is true because r_n divides r_{n-1} as seen from the $(n+1)^{\text{th}}$ equation. Thus, we see that r_n, which is the last divisor (or the last non-zero remainder) in the Euclid's algorithm, is the HCF of a and b. $\square$

Example 23. Using Euclid's algorithm, determine whether the numbers 20 and 12 are coprime or not.

Solution: Euclid's algorithm for the two integers comprises of the following equalities:

$$\begin{aligned} 20 &= 12 \times 1 + 8, \\ 12 &= 8 \times 1 + 4, \\ 8 &= 4 \times 2 + 0. \end{aligned}$$

Here, the last divisor is 4 and hence the HCF of 20 and 12 is 4. Since the HCF is not 1, the numbers 20 and 12 are not coprime.

Example 24. Find the HCF of 50, 60 and 70. Given that the LCM of the numbers is 2100, determine whether the product of the numbers is equal to the product of their HCF and LCM.

Solution: We have

$$\begin{aligned} 60 &= 50 \times 1 + 10, \\ 50 &= 10 \times 5 + 0. \end{aligned}$$

Therefore, the HCF $(50,60) = 10$. Now,

$$70 = 10 \times 7 + 0.$$

Therefore, the HCF $(10,70) = 10$.
Thus, $(50,60,70) = ((50,60),70) = (10,70) = 10$.

Now, the product of the numbers $= 50 \times 60 \times 70 = 210000$, and the product of their HCF and LCM $= 10 \times 2100 = 21000$. Hence, the product of the numbers is not equal to the product of their HCF and LCM.

Exercise 25. What is the HCF of two positive integers a and b if b is a factor of a? (Answer: b.)

Exercise 26. Find the HCF of the smallest prime number greater than 10 and the largest composite number less than 50. (Answer: 1.)

Exercise 27. The HCF of a prime number and a composite number is either 1 or the prime number. Justify.

Example 28. Prove that the fraction $\dfrac{21n+4}{14n+3}$ is irreducible (i.e., is in lowest terms) for every natural number n. **(IMO 1959)**

Solution: A fraction $\dfrac{a}{b}$ is irreducible if $\gcd(a,b) = 1$. We shall use Euclid's algorithm to show that $(21n+4, 14n+3) = 1 \ \forall\ n \in \mathbb{N}$. We have

$$\begin{aligned}21n+4 &= (14n+3)\times 1 + (7n+1),\\ 14n+3 &= (7n+1)\times 2 + 1,\\ 7n+1 &= 1\times(7n+1)+0.\end{aligned}$$

Here, the last divisor is 1 and hence $(21n+4, 14n+3) = 1 \ \forall\ n \in \mathbb{N}$. Consequently $\dfrac{21n+4}{14n+3}$ is irreducible for every natural number n. □

Remark: If d is a common divisor of a and b, then d divides $ax+by$ for any integers x and y. Example 28 can be solved by using this result. If $(21n+4, 14n+3) = d$, then d divides $-2(21n+4)+3(14n+3)$, i.e., d divides 1. Hence, $d = 1$.

Exercise 29. Prove that the fraction $\dfrac{12n+1}{30n+2}$ is irreducible for every natural number n.

Example 30. Prove that $(2^n-1, 2^n+1) = 1$ for any positive integer n.

Solution: If $n = 1$, then clearly $(2^n-1, 2^n+1) = (1,3) = 1$.
For $n \geq 2$, we have

$$\begin{aligned}2^n+1 &= (2^n-1)\times 1 + 2,\\ 2^n-1 &= 2\times 2^{n-1} - 1 = 2\times 2^{n-1} - 2 + 1 = 2\times\left(2^{n-1}-1\right)+1,\\ 2 &= 1\times 2+0.\end{aligned}$$

Hence, $(2^n-1, 2^n+1) = 1$ for $n \geq 2$. This completes the proof. □

Remark: If $d = (2^n-1, 2^n+1)$, then d divides $(2^n+1)-(2^n-1)$, i.e., d divides 2. But 2 does not divide 2^n+1 for any positive integer n. So, d cannot be 2. Consequently $d = 1$.

Exercise 1.1

1. Using Euclid's algorithm find the HCF of

(i) 1240 and 1984,

(ii) 348 and 504,

(iii) 986 and 899,

(iv) 4216 and 1240,

(v) 10605 and 5256,

(vi) 10005 and 9269.

Solution:

(i) Euclid's algorithm for the two integers 1240 and 1984 comprises of the following equalities:

$$\begin{aligned} 1984 &= 1240 \times 1 + 744, \\ 1240 &= 744 \times 1 + 496, \\ 744 &= 496 \times 1 + 248, \\ 496 &= 248 \times 2 + 0. \end{aligned}$$

The last divisor is 248 and hence the required HCF is 248.

(ii) The two given integers are 348 and 504. By Euclid's algorithm, we have

$$\begin{aligned} 504 &= 348 \times 1 + 156, \\ 348 &= 156 \times 2 + 36, \\ 156 &= 36 \times 4 + 12, \\ 36 &= 12 \times 3 + 0. \end{aligned}$$

The last divisor is 12 and hence the required HCF is 12.

(iii) The two given integers are 986 and 899. By Euclid's algorithm, we have

$$\begin{aligned} 986 &= 899 \times 1 + 87, \\ 899 &= 87 \times 10 + 29, \\ 87 &= 29 \times 3 + 0. \end{aligned}$$

The last divisor is 29 and hence the required HCF is 29.

(iv) The two given integers are 4216 and 1240. By Euclid's algorithm, we have

$$\begin{aligned} 4216 &= 1240 \times 3 + 496, \\ 1240 &= 496 \times 2 + 248, \\ 496 &= 248 \times 2 + 0. \end{aligned}$$

The last divisor is 28 and hence the required HCF is 248.

(v) The two given integers are 10605 and 5256. By Euclid's algorithm, we have

$$\begin{aligned} 10605 &= 5256 \times 2 + 93, \\ 5256 &= 93 \times 56 + 48, \\ 93 &= 48 \times 1 + 45, \\ 48 &= 45 \times 1 + 3, \\ 45 &= 3 \times 15 + 0. \end{aligned}$$

The last divisor is 3 and hence the required HCF is 3.

(vi) The two given integers are 10005 and 9269. By Euclid's algorithm, we have

$$\begin{aligned} 10005 &= 9269 \times 1 + 736, \\ 9269 &= 736 \times 12 + 437, \\ 736 &= 437 \times 1 + 299, \\ 437 &= 299 \times 1 + 138, \\ 299 &= 138 \times 2 + 23, \\ 138 &= 23 \times 6 + 0. \end{aligned}$$

The last divisor is 23 and hence the required HCF is 23.

2. Show that the product of two consecutive integers is divisible by 2.

Solution: Let a, $a+1$ be the two consecutive integers. Then a is of the form $2q$ or $2q+1$ for some integer q.

If $a = 2q$, then $a(a+1) = 2q(2q+1)$, which is divisible by 2.
If $a = 2q+1$, then $a(a+1)$

$$\begin{aligned} &= (2q+1)(2q+1+1) \\ &= 2(2q+1)(q+1), \text{ which is divisible by 2.} \end{aligned}$$

Thus, the product $a(a+1)$ is always divisible by 2. $\square$

3. Show that the product of two consecutive even integers is divisible by 8.

Solution: Let $2a$ and $2a+2$ be the two consecutive even integers where a is some integer. The integer a is of the form $2q$ or $2q+1$ for some integer q.

If $a = 2q$, then $2a(2a+2) = 4a(a+1)$

$$\begin{aligned} &= 4(2q)(2q+1) \\ &= 8q(2q+1), \text{ which is divisible by 8.} \end{aligned}$$

If $a = 2q+1$, then
$$\begin{aligned} 2a(2a+2) &= 4a(a+1) \\ &= 4(2q+1)(2q+1+1) \\ &= 4(2q+1)\{2(q+1)\} \\ &= 8(2q+1)(q+1), \text{ which is divisible by 8.} \end{aligned}$$

Thus, $2a(2a+2)$ is divisible by 8 for any integer a. □

4. Show that every integer is of the form $4q$, $4q+1$, $4q+2$ or $4q-1$.

Solution: By Euclid's division lemma, for any integer a, we have

$$a = 4k + r, \text{ where } 0 \le r < 4 \text{ and } k, r \text{ are integers.}$$

If $r = 0, 1$ or 2, then a is of the form $4q$, $4q+1$ or $4q+2$.
If $r = 3$, then $a = 4k+3 = 4(k+1)-1$, in which a is of the form $4q-1$.
Thus, a is of the form $4q$, $4q+1$, $4q+2$ or $4q-1$. □

5. Show that the product of three consecutive integers is divisible by 6.

Solution: Let a, $a+1$ and $a+2$ be the three consecutive integers. The integer a is of the form $2q$ or $2q+1$. If $a = 2q$, then

$$a(a+1)(a+2) = 2q(2q+1)(2q+2), \text{ which is divisible by 2.}$$

If $a = 2q+1$, then

$$\begin{aligned} a(a+1)(a+2) &= (2q+1)(2q+2)(2q+3) \\ &= 2(2q+1)(q+1)(2q+3), \text{ which is divisible by 2.} \end{aligned}$$

Thus, $a(a+1)(a+2)$ is divisible by 2 for any integer a.

Also, the integer a is of the form $3q$, $3q+1$ or $3q+2$. If $a = 3q$, then

$$a(a+1)(a+2) = 3q(3q+1)(3q+2), \text{ which is divisible by 3.}$$

If $a = 3q+1$, then

$$a(a+1)(a+2) = 3(3q+1)(3q+2)(q+1), \text{ which is divisible by 3.}$$

If $a = 3q+2$, then

$$a(a+1)(a+2) = 3(3q+2)(q+1)(3q+4), \text{ which is divisible by 3.}$$

Thus, $a(a+1)(a+2)$ is divisible by 3 for any integer a.

It is observed that $a(a+1)(a+2)$ is divisible by both 2 and 3 for any integer a. Hence, $a(a+1)(a+2)$ is divisible by 6. □

Remark: Q.5 can be solved using the fact that any integer a is of the form $6q+r$, where $0 \le r < 6$. See the solution to Q.11, page 13.

6. Show that the square of an odd integer is of the form $8k+1$.

   Solution: Let $2a+1$ be any odd integer where a is some integer. Then $(2a+1)^2 = 4a^2+4a+1 = 4a(a+1)+1$. The integer a is of the form $2q$ or $2q+1$.

   If $a = 2q$, then
$$\begin{aligned}(2a+1)^2 &= 4(2q)(2q+1)+1\\ &= 8q(2q+1)+1\\ &= 8k+1, \text{ where } k = q(2q+1) \in \mathbb{Z}.\end{aligned}$$
   And if $a = 2q+1$, then
$$\begin{aligned}(2a+1)^2 &= 4(2q+1)(2q+2)+1\\ &= 8(2q+1)(q+1)+1\\ &= 8k+1, \text{ where } k = (2q+1)(q+1) \in \mathbb{Z}.\end{aligned}$$

   Thus, the square of an odd integer is of the form $8k+1$. □

7. If a is divisible by neither 2 nor 3, show that a^2-1 is divisible by 24.

   Solution: The integer a is of the form $6q+r$, where $0 \le r < 6$. If $r = 0, 2, 3$ or 4, then a is divisible by 2 or 3. Since a is divisible neither by 2 nor by 3, we have $r = 1$ or 5. That is, a is of the form $6q+1$ or $6q+5$.

   If $a = 6q+1$, then
$$\begin{aligned}a^2-1 &= (6q+1)^2-1\\ &= (6q+1-1)(6q+1+1)\\ &= 12q(3q+1).\end{aligned}$$
   If q is even, then clearly a^2-1 is divisible by 24.
   If q is odd, then $3q+1$ is even and hence a^2-1 is divisible by 24.
   If $a = 6q+5$, then
$$\begin{aligned}a^2-1 &= (6q+5)^2-1\\ &= (6q+5-1)(6q+5+1)\\ &= 12(3q+2)(q+1).\end{aligned}$$
   If q is even, then $(3q+2)$ is even and hence a^2-1 is divisible by 24.
   If q is odd, then $q+1$ is even and hence a^2-1 is divisible by 24.

   So, a^2-1 is divisible by 24 if a is divisible neither by 2 nor by 3. □

8. Show that any square number cannot be put in the form $4k+2$.

   Solution: Every integer n is of the form $2a$ or $2a+1$.

If $n = 2a$, then $n^2 = 4a^2 = 4q$, where $q = a^2$ is an integer. If $n = 2a+1$, then $n^2 = 4a^2 + 4a + 1 = 4(a^2 + a) + 1 = 4q + 1$, where $q = a^2 + a$ is an integer.

Thus, we see that n^2 is of the form $4q$ or $4q + 1$. But a number of the form $4q$ or $4q + 1$ cannot be put in the form $4k + 2$.

$$\left[\begin{array}{l} 4q = 4k + 2 \implies 2(q - k) = 1 \implies \text{2 divides 1, impossible.} \\ 4q + 1 = 4k + 2 \implies 4(q - k) = 1 \implies \text{4 divides 1, impossible.} \end{array}\right]$$

Hence, n^2 cannot be put in the form $4k + 2$. □

Remark: See example 37, page 15 for another proof of Q.8.

9. Show that any square number is of the form $3n$ or $3n + 1$.

Solution: We know that any integer is of the form $3k$, $3k+1$, or $3k+2$. Now, we have

$$\begin{aligned} (3k)^2 &= 3(3k^2) = 3n, \text{ where } n = 3k^2, \\ (3k+1)^2 &= 9k^2 + 6k + 1 \\ &= 3(3k^2 + 2k) + 1 \\ &= 3n + 1, \text{ where } n = 3k^2 + 2k, \\ (3k+2)^2 &= 9k^2 + 12k + 4 \\ &= 3(3k^2 + 4k + 1) + 1 \\ &= 3n + 1, \text{ where } n = 3k^2 + 4k + 1. \end{aligned}$$

Thus, any square number is of the form $3n$ or $3n + 1$. □

10. Show that one of three consecutive odd integers is a multiple of 3.

Solution: Let a, $a + 2$ and $a + 4$ be the three consecutive odd integers.
The integer a is of the form $3q$, $3q + 1$ or $3q + 2$.
If $a = 3q$, then clearly a is a multiple of 3.
If $a = 3q + 1$, then $a + 2 = 3q + 3 = 3(q + 1)$ is a multiple of 3.
If $a = 3q + 2$, then $a + 4 = 3q + 6 = 3(q + 2)$ is a multiple of 3.
Thus, one of a, $a + 2$ and $a + 4$ is a multiple of 3. □

Remark: We can also take the three consecutive odd numbers to be $2a + 1$, $2a + 3$, $2a + 5$, where a is an integer. An argument similar to that of the solution to Q.10 can be given to show that one of three consecutive *even* integers is a multiple of 3. Also, note that for any integer a, the numbers $2a$, $2a + 2$, $2a + 4$ are three consecutive even numbers.

11. Show that the product of any three consecutive even integers is divisible by 48.

    Solution: Let $2a$, $2a+2$ and $2a+4$ be the three consecutive even integers where $a \in \mathbb{Z}$. Now, $2a(2a+2)(2a+4) = 8a(a+1)(a+2)$. The integer a is of the form $6q+r$, where $r = 0, 1, 2, 3, 4$ or 5.

    If $a = 6q$, then $\quad 8a(a+1)(a+2) = 8(6q)(6q+1)(6q+2) = 48k,$

    where $k = q(6q+1)(6q+2)$.

    If $a = 6q+1$, then $\quad 8a(a+1)(a+2) = 8(6q+1)(6q+2)(6q+3) = 48k,$

    where $k = (6q+1)(3q+1)(2q+1)$.

    If $a = 6q+2$, then $\quad 8a(a+1)(a+2) = 8(6q+2)(6q+3)(6q+4) = 48k,$

    where $k = 2(3q+1)(2q+1)(3q+2)$.

    If $a = 6q+3$, then $\quad 8a(a+1)(a+2) = 8(6q+3)(6q+4)(6q+5) = 48k,$

    where $k = (2q+1)(3q+2)(6q+5)$.

    If $a = 6q+4$, then $\quad 8a(a+1)(a+2) = 8(6q+4)(6q+5)(6q+6) = 48k,$

    where $k = 2(3q+2)(6q+5)(q+1)$.

    If $a = 6q+5$, then $\quad 8a(a+1)(a+2) = 8(6q+5)(6q+6)(6q+7) = 48k,$

    where $k = (6q+5)(q+1)(6q+7)$.

    Thus, $2a(2a+2)(2a+4) = 8a(a+1)(a+2) = 48k$ for some $k \in \mathbb{Z}$ and hence $2a(2a+2)(2a+4)$ is divisible by 48. □

    Remark: To solve Q.11, we can also show that $a(a+1)(a+2) = 6k$ for some integer k as in the solution to Q.5 on page 10.

Chapter 2

Polynomials

When you have eliminated all which is impossible, then whatever remains, however improbable, must be the truth.

— Arthur Conan Doyle, *The Case-Book of Sherlock Holmes*

A polynomial is an expression consisting of variables and coefficients, that involves addition, subtraction, multiplication, and non-negative integer exponents of variables. An example of single variable polynomial is $x^3 - 2x^2 - x + 2$. In this chapter, we shall discuss various properties of polynomials.

Division Algorithm for Polynomials

Definition 2.1 (**Degree of a polynomial**). If $p(x)$ is a polynomial in x, the highest exponent of x in $p(x)$ is called the degree of $p(x)$.

A polynomial is called **linear**, **quadratic**, **cubic**, **quartic** (**biquadratic**), or **quintic** according as its degree is one, two, three, four or five respectively.

Definition 2.2 (**Monic polynomial**). A polynomial $p(x)$ is said to be monic if the coefficient of the highest degree term in $p(x)$ is 1.

If $p(x)$ is a polynomial in x, and if k is any number, then the value obtained by replacing x by k in $p(x)$, is called **the value of** $p(x)$ **at** $x = k$, and is denoted by $p(k)$.

Definition 2.3 (**Zero of a polynomial**). A real number a is said to be a zero or a root of a polynomial $p(x)$ if $p(a) = 0$.

Geometrical meaning of the zeroes of a polynomial: The zeroes of a polynomial $p(x)$ are precisely the x-coordinates of the points where the graph representing $y = p(x)$ intersects the x-axis. The graph of a linear polynomial $ax + b, a \neq 0$, is a straight line. It intersects the x-axis at exactly one point. So, a linear polynomial has exactly one zero. The graph of a quadratic polynomial $ax^2 + bx + c, a \neq 0$, is a parabola. It intersects the x-axis at atmost 2 points. So, a quadratic polynomial has at most 2 zeroes. In general, given a polynomial $p(x)$ of degree n, the graph of $y = p(x)$ intersects the x-axis at atmost n points. So, a polynomial $p(x)$ of degree n has atmost n zeroes. The following figure shows the graph of $y = x^3 - 2x^2 - x + 2$ intersecting the x-axis at three distinct points.

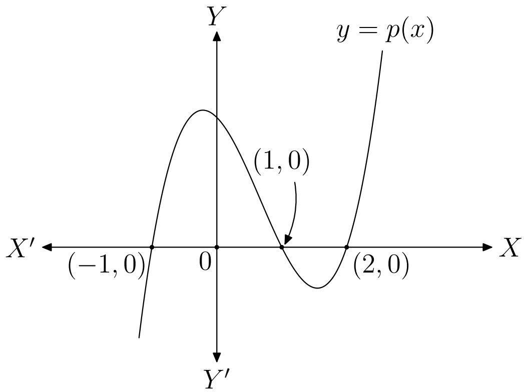


Figure 2.1: Graph of $y = p(x) = x^3 - 2x^2 - x + 2$.

Remark: The zeroes we have discussed above are the zeroes which are real. In the complex number system (see page 154), a polynomial of degree n has exactly n zeroes (counting multiplicities).

Long division process of polynomials:

Let us discuss the long division process of polynomials with the help of an example. Consider the division of $4 + 2x^2 + 3x$ by $2 + x$. We carry out the division by means of the following steps.

$$
\begin{array}{r|l}
 & \phantom{2x^2 + {}} 2x - 1 \\
\hline
x + 2 & 2x^2 + 3x + 4 \\
 & \underline{2x^2 + 4x} \\
 & -x + 4 \\
 & \underline{-x - 2} \\
 & \phantom{2x^2 + 3x + {}} 6
\end{array}
$$

Step 1. We arrange the terms of the dividend and the division in the descending order of their degrees. (If any term is missing in the dividend, a zero may be used to fill in the missing term.) The dividend is $2x^2 + 3x + 4$ and the divisor is $x + 2$.

Step 2. We divide the first term of the dividend by the first term of the divisor, i.e., we divide $2x^2$ by x and we get $2x$. $2x$ is the first term of the quotient.

Step 3. We multiply the divisor $x+2$ by $2x$ (the first term of the quotient) and obtain the product $2x^2+4x$. We subtract this product $2x^2+4x$ from the dividend $2x^2+3x+4$ and we get the remainder $-x+4$.

Step 4. We treat the remainder $-x+4$ as the new dividend, keeping the divisor the same. We divide the first term $-x$ of the new dividend by the first term of the divisor and obtain -1. -1 is the second term of the quotient.

Step 5. We multiply the divisor $x+2$ by -1 (the second term of the quotient) and subtract the product $-x-2$ from the dividend $-x+4$. This gives 6 as the remainder which will be the new dividend for the next step.

Note that steps 4 and 5 are repetitions of the steps 2 and 3 with a new dividend. The process continues until the remainder is zero or the degree of the remainder (the new dividend) is less than that of the divisor. In our present case, the degree of the new dividend 6 is less than the degree of the divisor $x+2$ and so we stop the process here. The last remainder (or the new dividend at the last stage) 6 is the remainder of the division of $2x^2+3x+4$ by $x+2$. The quotient is $2x-1$ (the sum of the quotient terms obtained in steps 2 and 4). We see that $2x^2+3x+4=(x+2)(2x-1)+6$, i.e., dividend = divisor × quotient + remainder.

Theorem 2.1 (**Division algorithm for polynomials**). *If $p(x)$ and $d(x)$ are any two polynomials with $d(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that $p(x)=d(x)\times q(x)+r(x)$, where either $r(x)=0$ or degree of $r(x) <$ degree of $d(x)$.*

Here, $q(x)$ is called the quotient and $r(x)$ is called the remainder.

Remark: In Chapter 1, we have defined the HCF and the LCM of two numbers. In a similar way, we can define the HCF and the LCM of two polynomials. We have also discussed the Euclid's algorithm for finding the HCF of two numbers. A similar algorithm can be used for finding the HCF of two polynomials.

Exercise 1. Find the quotient $q(x)$ and the remainder $r(x)$ when the polynomial $p(x)$ is divided by the polynomial $d(x)$ and verify the division algorithm in each of the following:

(i) $p(x)=3x^3-5x^2+10x+5$, $d(x)=3x+1$,

(ii) $p(x)=x^3-2x^2-12$, $d(x)=3-x$,

(iii) $p(x)=2x^3-9x^2+25$, $d(x)=2x-5$,

(iv) $p(x)=x^3-12x^2-22$, $d(x)=x^2-2x+1$,

(v) $p(x)=4x^4+3x^3-2x+6$, $d(x)=x^2+x+2$.

Chapter 5

Quadratic Equations

> *"They're not tough, though. I merely take advantage of the blind spots created when students assume too much. And they usually assume too much."*
> *"Blind spots?"*
> *"For instance, I give them a question that looks like a geometry problem, but is in fact an algebra problem. If all they've done is memorize the problem sheets in their books–"*
>
> — Keigo Higashino, *The Devotion of Suspect X*

In this chapter, we shall discuss various methods of finding the roots of a quadratic equation. We shall also discuss the relationship between the roots and the coefficients of a quadratic equation, and the formation of a quadratic equation when its roots are given.

Solution of Quadratic Equations

Definition 5.1. An equation of the form $ax^2+bx+c=0$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, is called a quadratic equation with real coefficients in the variable x.

In fact, any equation of the form $p(x) = 0$, where $p(x)$ is a polynomial of degree 2, is a quadratic equation. When the terms of $p(x)$ are written in descending order of their degrees, we obtain an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Therefore, $ax^2 + bx + c = 0$, where $a \neq 0$, is called the standard form of a quadratic equation.

Definition 5.2 (Roots of a quadratic equation). A real number α is called a root of the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$, if $a\alpha^2 + b\alpha + c = 0$.

We also say that $x = \alpha$ is a solution of the quadratic equation or that α satisfies the quadratic equation. Note that a root of the quadratic equation $ax^2 + bx + c = 0$ is a zero of the polynomial $ax^2 + bx + c$ and vice-versa. A quadratic equation cannot have more than two roots.

Exercise 1. Check whether the following are quadratic equations:
(i) $x^3 + 2015 = (x-1)^3$, **(ii)** $2x^2 + 3x - 1 = (2x+1)(x+5)$.
(Answer: (i) Quadratic. (ii) Not quadratic.)

Quadratic equations can be solved by (i) the method of factorisation, or by (ii) the method of completing the square, also known as the Hindu method or the Sreedharacharya's method.

Solution of a quadratic equation by factorisation:

In this method, we find the roots of the quadratic equation $ax^2+bx+c = 0$, $a \neq 0$, by factorising the polynomial $ax^2 + bx + c$ into two linear factors and then equating each factor to 0 (using corollary 1.13, page 29).

Exercise 2. Solve by the method of factorisation:
(a) $x^2 - 3x - 10 = 0$, (Answer: -2, 5.)
(b) $2x^2 - 5x + 3 = 0$, (Answer: 1, 3/2.)
(c) $100x^2 - 20x + 1 = 0$. (Answer: 1/10, 1/10.)

Exercise 3. Solve: $\dfrac{4}{x} - 3 = \dfrac{5}{2x+3}$. (Answers: -2, 1.)

Solution of a quadratic equation by completing the square:

Consider the quadratic equation $ax^2 + bx + c = 0$, $a \neq 0$. Dividing throughout by a, we get

$$
\begin{aligned}
& x^2 + \frac{b}{a}x + \frac{c}{a} = 0 \\
\Longrightarrow\ & x^2 + 2 \times x \times \frac{b}{2a} + \left(\frac{b}{2a}\right)^2 - \left(\frac{b}{2a}\right)^2 + \frac{c}{a} = 0 \\
\Longrightarrow\ & \left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2} \\
\Longrightarrow\ & x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a} \\
\Longrightarrow\ & x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.
\end{aligned}
$$

Note: Completion of the square can also be done after multiplying the given equation throughout by $4a$.

Quadratic formula:

The roots of the quadratic equation $ax^2 + bx + c = 0$, where $a \neq 0$, are

$$\alpha = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \quad \text{and} \quad \beta = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Do you know? Évariste Galois was born on October 25, 1811, near Paris. The first eleven years of his life were happy. Galois, at the age of sixteen, attempted to enter the prestigious École Polytechnique, but failed in the entrance examination. Commenting on his failure, Terquem remarked, "A candidate of superior intelligence is lost with an examiner of inferior intelligence." After this, things started to go very bad for Galois. On April 2, 1829, Galois' father committed suicide. He once again tried to enter the École Polytechnique, but again failed under some rather controversial circumstances. In one last desperate effort to gain recognition, in 1831, he had sent a memoir on the general solution of equations to the Academy of Sciences. Moreover, Galois' proof was, to say least, sketchy. Poisson, after reading Galois' memoir, remarked

> We have made every effort to understand Mr. Galois' proof. His arguments are not clear enough, nor developed enough, for us to be able to judge their correctness...

Galois' paper was rejected for publication. In May 1832, Galois had a brief love affair with a young woman. He broke of the affair on May 14, and this appears to be the cause of subsequent duel that proved fatal to Galois. Galois died on May 31, 1832 at the age of 20. Fourteen years later, in 1846, Galois' work was finally published. What he proved in his paper was that for any $n \geq 5$, there is no algebraic formula, involving only the four basic arithmetic operations and the taking of roots, that gives the solutions to any polynomial equation of degree n.

Exercise 4. Solve by method of completing the square:

$$4x^2 - 2(a^2 + b^2)x + a^2b^2 = 0. \qquad \left(\text{Answers: } a^2/2,\ b^2/2.\right)$$

Exercise 5. Solve by using quadratic formula:

$$abx^2 + (b^2 - ac)x - bc = 0 \qquad (ab \neq 0). \qquad (\text{Answers: } c/b,\ -b/a.)$$

Exercise 6. Factorise the quadratic polynomial $ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$ and $a \neq 0$, by using quadratic formula. State the conditions under which the polynomial can or cannot be factorised over $\mathbb{R}$.

Chapter 9

Construction

All ornament should be based upon a geometrical construction.

— Owen Jones, *The Grammar of Ornament*

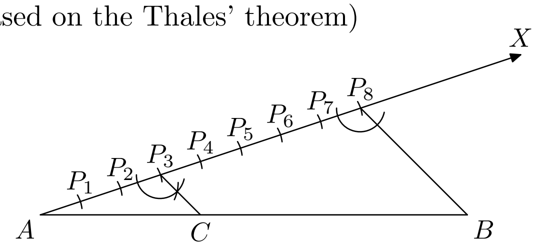
As far as possible, only two geometrical instruments, namely a ruler and a compass, will be used in geometrical construction. The analysis part of a construction is given only to reveal the clues leading to the construction process and may be omitted.

Division of a Line Segment in a given Ratio

Recall that a point C on a line segment AB is said to divide AB *internally* in the ratio $m : n$ if $AC : CB = m : n$. There are two methods for dividing a given line segment internally in a given ratio, one based on the Thales' theorem (basic proportionality theorem) and the other based on the property of similar triangles.

Example 1. Draw any line segment and divide it internally in the ratio $3 : 5$. Write the steps of construction. Also, justify the construction.

Solution: (Based on the Thales' theorem)



Steps of construction:

(1) Draw a line segment AB of any length.

(2) Draw a ray AX inclined to AB at an acute angle.

(3) Mark eight $(= 3 + 5)$ points $P_1, P_2, \ldots, P_8$ on AX such that $AP_1 = P_iP_{i+1}$ for all $i = 1, 2, \ldots, 7$.

(4) Join P_8B and through the point P_3 draw a line parallel to P_8B meeting AB at C.

Then C is the point on AB such that $AC : CB = 3 : 5$.

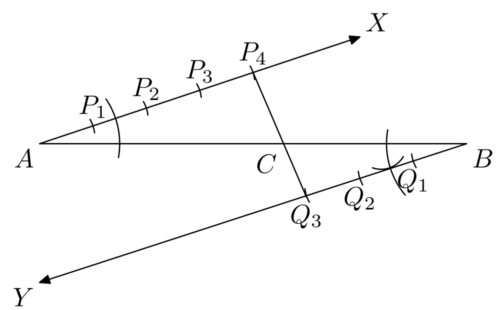
Justification: In $\triangle ABP_8$, by construction, we have $P_3C \| P_8B$. So, by Thales' theorem, we have

$$AC : CB = AP_3 : P_3P_8.$$

But, by construction, we have $AP_3 : P_3P_8 = 3 : 5$.
Hence, $AC : CB = 3 : 5$. □

Example 2. Draw a line segment AB and divide it in the ratio $4 : 3$. Write the steps of construction. Also, give the justification of the construction.

Solution: (Based on the property of the similarity of triangles)



Steps of construction:

(1) Draw a line segment AB of any length.

(2) Draw a ray AX inclined to AB at an acute angle.

(3) Draw a ray BY parallel to AX so that $\angle ABY = \angle BAX$.

(4) Mark points P_i's $(i = 1, 2, 3, 4)$ on AX and Q_j's $(j = 1, 2, 3)$ on BY such that $AP_1 = P_iP_{i+1} = BQ_1 = Q_jQ_{j+1}$ for all $i = 1, 2, 3$ and $j = 1, 2$.

(5) Join P_4Q_3 intersecting AB at C.

Then C is the point on AB such that $AC : CB = 4 : 3$.

Justification: In $\triangle ACP_4$ and $\triangle BCQ_3$,

$$\angle P_4AC = \angle Q_3BC \quad (\because AP_4 \| BQ_3)$$
$$\text{and } \angle ACP_4 = \angle BCQ_3 \quad \text{(vertically opposite angles).}$$

By AA similarity criterion, we have $\triangle ACP_4 \sim \triangle BCQ_3$.

$$\therefore\ AC : CB = AC : BC = AP_4 : BQ_3 = 4 : 3. \qquad \square$$

Note: In the following, by construction, we mean construction by using only a straightedge (with no markings) and a compass.

(i) (*Doubling the Cube*) It is impossible to construct a cube with precisely twice the volume of a given cube.

(ii) (*Trisecting an Angle*) It is impossible to trisect any given angle θ.

(iii) (*Squaring the Circle*) It is impossible to construct a square whose area is precisely the area of a given circle.

Exercise 3. To divide a line segment AB in the ratio $m : n$ (where m and n are positive integers), first a ray AX is drawn so that $\angle BAX$ is an acute angle and then at equal distances points are marked on the ray AX. What is the minimum number of these points? (Answer: $m + n$.)

Exercise 4. To divide a line segment AB in the ratio $7 : 9$, a ray AX is drawn first such that $\angle BAX$ is an acute angle and then points $P_1, P_2, P_3, \ldots$ are located at equal distances on the ray AX. Which point is then joined to the point B? (Answer: P_{16}.)

Exercise 5. Construct a triangle in which its perimeter is given and its three sides are in the ratio $2 : 3 : 4$.

Hint: Draw a line segment PQ equal in length to that of the perimeter of the triangle. Divide PQ in the ratio $2 : 3 : 4$ so that $PB : BC : CQ = 2 : 3 : 4$. With centre B and radius BP, draw an arc. With centre C and radius CQ, draw an arc intersecting the previous arc at A. Then ABC is the required triangle.

Exercise 6. Divide a line segment of length 6 cm in the ratio $\dfrac{2}{\sqrt{3}} : \sqrt{3}$.

Hint: The given ratio is equal to $2 : 3$.

Exercise 7. Two line segments AB and AC include an angle of 60° where $AB = 5$ cm and $AC = 7$ cm. Locate points P and Q on AB and AC, respectively such that $AP = \dfrac{3}{4}AB$ and $AQ = \dfrac{1}{4}AC$. Join P and Q and measure the length PQ. (Answer: 3.25 cm.)

Chapter 13

Statistics

"Data! Data! Data!" he cried impatiently. "I can't make bricks without clay."

— Arthur Conan Doyle, *The Adventure of Sherlock Holmes*

We know that statistics deals with collection, compilation, analysis and interpretation of data. Statistics gives information which are of representative nature and do not pertain to information on particular individuals. In this chapter, we discuss the measures of central tendency and the measures of location for grouped data.

Measures of Central Tendency

Definition 13.1 (Measure of central tendency). A measure of central tendency is the value of the variate around which the other variate values are supposed to cluster.

Recall that when the number of observations is very large, data are condensed into groups called *classes* or *class intervals*. The boundaries of a class are called *class limits*. The half of the sum of the lower limit and the upper limit of a class is called the *class mark* or *mid value of the class*. The number of observations lying in a class is called the *frequency of the class*.

In a grouped frequency distribution, all the frequencies are distributed in different classes. The sum of all the frequencies of all the classes is called the *size of the sample or the population*.

For a continuous grouped frequency distribution in which the upper limit of a class is the lower limit of the following class, usually an item equal to the upper limit of a class is excluded from that class but an item equal to the lower limit of a class is included in that class.

Remark: The frequency of each class interval is assumed to be centred around its mid value. So, the mid value (or class mark) of each class is chosen to represent the observations falling in that class.

Do you know? One day, in 1939, a graduate student at the University of California, Berkeley arrived late for a class. He found two problems on the blackboard and assumed them to be homework problems. The problems seemed to be a little harder than usual, but a few days later he submitted the complete solutions for the problems. About six weeks later, one Sunday morning, he received a visit from the professor, who informed him that he had prepared one of his solutions for publication in a mathematical journal. The problems he had solved were in fact two famous unsolved problems in statistics. The professor wrote the problems as examples of unsolved problems. The student was none other than George Dantzig, the man who later formulated the simplex method in linear programming. Linear programming is a mathematical technique for optimization of an outcome (such as maximizing profit and minimizing cost).

Definition 13.2. For a grouped frequency distribution having $x_1, x_2, \ldots, x_n$ as mid values of the classes with respective frequencies $f_1, f_2, \ldots, f_n$,

(i) the **weighted arithmetic mean** is the quantity $\overline{x}$ given by

$$\overline{x} = \frac{x_1 f_1 + x_2 f_2 + \cdots + x_n f_n}{f_1 + f_2 + \cdots + f_n} = \frac{1}{N}\sum_{i=1}^{n} x_i f_i, \quad \text{where } N = \sum_{i=1}^{n} f_i,$$

(ii) the **weighted geometric mean**, denoted by G, is

$$G = \left(x_1^{f_1} x_2^{f_2} \cdots x_n^{f_n}\right)^{\frac{1}{f_1+f_2+\cdots+f_n}} = \left(\prod_{i=1}^{n} x_i^{f_i}\right)^{\frac{1}{N}}, \quad \text{where } N = \sum_{i=1}^{n} f_i,$$

(iii) the **weighted harmonic mean**, denoted by H, is given by

$$\frac{f_1 + \cdots + f_n}{H} = \sum_{i=1}^{n} \frac{f_i}{x_i} \quad \textbf{or} \quad \frac{1}{H} = \frac{1}{N}\sum_{i=1}^{n}\frac{f_i}{x_i}, \quad \text{where } N = \sum_{i=1}^{n} f_i.$$

Remark: $A \geq G \geq H$, where A is the arithmetic mean, G is the geometric mean and H is the harmonic mean.

Exercise 12. The following is the frequency distribution of marks obtained by 60 students in mathematics.

Marks	0 – 10	10 – 20	20 – 30	30 – 40	40 – 50	50 – 60	60 – 70
No. of students	4	8	11	15	12	7	3

(i) Find the mean, the median and the mode of the data given above. Compare and interpret the three measures of central tendency.

(ii) Find the lower and the upper quartiles, the 8^{th} decile and the 13^{th} percentile for the given data. Interpret these measures of location.

(iii) Draw both ogives for the given data. Determine the median, the 8^{th} decile and the 13^{th} percentile from the graph.

Hint: Mean = 34.33. Median = 34.67. Mode = 35.71. $Q_1 = 22.73$. $Q_3 = 45.83$. $D_8 = 48.33$. $P_{13} = 14.75$. On an average a student obtained the mean mark. About half the students scored marks less than the median mark and the other half scored marks more than the median mark. Maximum number of students scored near the modal mark.

Table 1. Cumulative frequency distribution of the less than type.

Marks obtained	Number of students
Less than 10	4
Less than 20	$4 + 8 = 12$
Less than 30	$12 + 11 = 23$
Less than 40	$23 + 15 = 38$
Less than 50	$38 + 12 = 50$
Less than 60	$50 + 7 = 57$
Less than 70	$57 + 3 = 60$

In Table 1 above, $10, 20, \ldots, 70$ are the upper limits of the respective classes. Plot the points $(10, 4)$, $(20, 12)$, etc., and join them by free hand to get the less than ogive (or cumulative frequency curve of the less than type).

Table 2. Cumulative frequency distribution of the more than type.

Marks obtained	Number of students
More than or equal to 0	60
More than or equal to 10	$60 - 4 = 56$
More than or equal to 20	$56 - 8 = 48$
More than or equal to 30	$48 - 11 = 37$
More than or equal to 40	$37 - 15 = 22$
More than or equal to 50	$22 - 12 = 10$
More than or equal to 60	$10 - 7 = 3$

In Table 2 above, $0, 10, \ldots, 60$ are the lower limits of the respective classes. Plot the points $(0, 60)$, $(10, 56)$, etc., and join them by free hand to get the more than ogive (or cumulative frequency curve of the more than type).

The less than and the more than ogives of the given data are shown below.

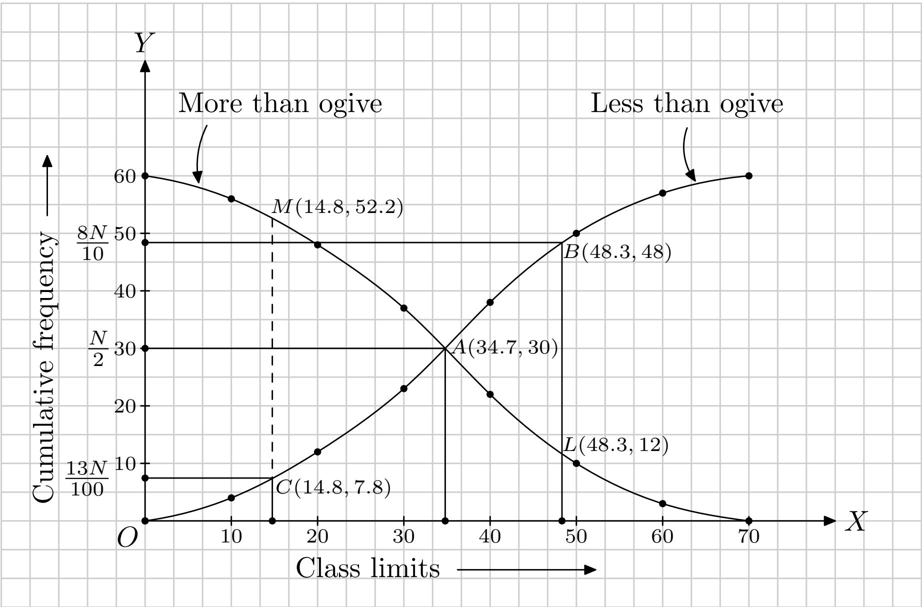


Figure 13.1: Ogives.

The less than ogive and the more than ogive of the given data intersect at a point A. Median is the x-coordinate of the point A. Mark points B and C on the less than ogive whose y-coordinates are $\frac{8N}{10}$ and $\frac{13N}{100}$ respectively. D_8 and P_{13} are the x-coordinates of the points B and C respectively. Also, D_8 and P_{13} are the x-coordinates of the points L and M (see figure above) on the more than ogive respectively. Note that the corresponding y-coordinates of the points L and M on the more than ogive are $\frac{(10-8)N}{10}$ and $\frac{(100-13)N}{100}$ respectively.

Exercise 13. The more than ogive curve and the less than ogive curve of a frequency distribution intersect each other at at the point $(30, 50)$.

(a) What is the median of the distribution? (Answer: 30.)

(b) What is the size of the population? (Answer: 100.)

(c) If the mean is 31, estimate the mode of the distribution. (Answer: 28.)

Exercise 14. If the mean and the median of a frequency distribution differs by 1.25, estimate the difference between the mean and the mode using Pearson's empirical formula. (Answer: 3.75.)

Exercise 15. Find the mean, the median, the mode, the lower and the upper quartiles, the 7^{th} decile and the 59^{th} percentile of the following distribution of daily savings of 106 workers.

Daily Savings (in ₹)	Number of workers
0.1 – 4.9	5
5.1 – 9.9	11
10.1 – 14.9	26
15.1 – 19.9	10
20.1 – 24.9	21
25.1 – 29.9	13
30.1 – 34.9	9
35.1 – 39.9	6
40.1 – 44.9	3
45.1 – 49.9	2

Draw the less than and the more than ogives for the given data. Determine the median, the lower quartile and the upper quartile from the graph.

Hint: Convert the given discontinuous class intervals into continuous ones by subtracting $\dfrac{5.1 - 4.9}{2} = 0.1$ from the lower limit and adding 0.1 to the upper limit of each class. Mean = 20.42. Median = 20.24. Mode = 12.42. $Q_1 = 12.02$. $Q_3 = 27.50$. $D_7 = 25.46$. $P_{59} = 22.51$.

Exercise 16. For the following frequency distribution, the mode is 26 and the median is 22.4. Find the values of the unknown entries a and b, and then calculate the mean of the distribution.

Class	Frequency
0 – 8	15
8 – 16	13
16 – 24	15
24 – 32	a
32 – 40	b
40 – 48	10

Hint: The modal class is 24 – 32 since 26 lies in this class. Using the formula for finding mode, we get $2a + b = 45$. The median class is 16 – 24 since 22.4 lies in this class. Using the formula for finding median, we get $a + b = 27$. Solving the equations, we get $a = 18$, $b = 9$. Mean = 22.3.

Exercise 17. Find the values of the unknown entries a, b, c, d, e, f, and hence find the mean, the median and the mode for the following frequency distribution.

Class	Frequency	Cumulative Frequency
$0-8$	15	a
$8-16$	b	28
$16-24$	15	c
$24-32$	d	61
$32-40$	e	70
$40-48$	10	f

Hint: $a = 15$, $b = 28 - a$, $c = 28 + 15$, $d = 61 - c$, $e = 70 - 61$, $f = 70 + 10$. Mean = 22.3. Median = 22.4. Mode = 26.

Exercise 18. Which measure of central tendency will be the most suitable in each of the following cases? Justify your answer in each case.

(1) To determine the productivity of a field using the data of the yield of the field for the past twenty years.

(2) To determine whether the literacy rate is the maximum in the age group 6 years to 14 years.

(3) To find the average of the marks obtained by the students in an examination.

(4) To find the average of the marks obtained by most of the students in an examination.

(5) To find the the mark above which only half the students scored in an examination.

(6) To find the typical productivity rate of workers.

(7) To find the average wage in a country.

(8) To find the most popular T.V. programme being watched.

(9) To determine the colour of the vehicle used by most of the people.

Hint: The measure of central tendency under study should possess the representative character of the data. Depending on the nature of the information that one is looking for, the appropriate measure of central tendency is to be fixed. Mean is suitable for (1), (3). Median is suitable for (5), (6), (7). There may be extreme values in (6) and (7). The mean is greatly affected by extreme values. So, rather than the mean, we take the median as a better measure of central tendency. Mode is suitable for (2), (4), (8), (9).

Exercise 13.1

1. The following are the numbers of children in a locality of 30 families.

$$7, 4, 0, 4, 2, 1, 2, 5, 3, 1, 4, 6, 2, 1, 4, 3, 2, 0, 1, 2, 5, 4, 2, 3, 2, 2, 1, 2, 1, 3.$$

Find the average number of children per family.

Solution: The average number of children per family

$$= \frac{\text{total number of children in the locality}}{\text{number of family}}$$

$$= \frac{7+4+0+4+2+1+2+5+3+1+6+2+1+4+3+2+0+1+2+5+4+2+3+2+2+1+2+1+3}{30}$$

$$= \frac{79}{30} = 2.63.$$

Remark: For an ungrouped data with variates $x_1, x_2, \ldots, x_n$, the arithmetic mean (AM), $\overline{x} = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

2. A shop dealing in electric goods makes the following record of T.V. sets sold during a particular year.

Month	No. of T.V. sets sold
January	14
February	17
March	16
April	12
May	7
June	6
July	8
August	9
September	6
October	14
November	15
December	18

Find the average number of T.V. sets sold per month.

Solution: The average number of T.V. sets sold per month

$$= \frac{\text{total number of T.V. sets sold in a year}}{\text{number of months in a year}}$$

$$= \frac{14 + 17 + 16 + 12 + 7 + 6 + 8 + 9 + 6 + 14 + 15 + 18}{12}$$

$$= \frac{142}{12} = 11.83.$$

3. The following is the record of weights of children at the time of their birth as maintained in a maternity ward during a particular year.

Weight of a child in kg	No. of children
2.5	12
3.0	175
3.5	156
3.8	42
4.0	15
4.2	5

Find the average (mean) weight of a child at the time of birth.

Solution: The given data is modified as below:

Weight of the children in kg (x_i)	No. of children (f_i)	$f_i x_i$
2.5	12	30
3.0	175	525
3.5	156	546
3.8	42	159.6
4.0	15	60
4.2	5	21
	$N = \sum_{i=1}^{6} f_i = 405$	$\sum_{i=1}^{6} f_i x_i = 1341.6$

So, the mean weight of a child at the time of birth is given by

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{6} f_i x_i = \frac{1}{405} \times 1341.6 = 3.31 \text{ kg}.$$

4. Five coins are simultaneously tossed 1000 times, and at each toss the number of heads was observed. The number of tosses during which $0, 1, 2, 3, 4, 5$ heads were obtained are shown in the table below. Find the number of heads per toss.

No. of heads per toss	No. of tosses
0	38
1	144
2	342
3	287
4	164
5	25

Chapter 14

Probability

Since in action it frequently happens that no delay is permissible, it is very certain that, when it is not in our power to determine what is true, we ought to act according to what is most probable.

— René Descartes, *Discourse on the Method*

Probability is the measure of chance of occurrence or non occurrence of an event. The definition of probability is given in three seemingly different forms, namely

1. the empirical or experimental definition,
2. the classical or mathematical or a priori definition of probability due to Pierre-Simon Laplace, and
3. the set theoretic or axiomatic or modern definition due to Andrey Kolmogorov.

The values of the probability of an event as calculated from these three seemingly different stand points converge to the same value as the number of trials becomes larger and larger. In this chapter, we discuss the classical definition.

Classical Definition of Probability

Definition 14.1 (Random or non-deterministic experiment). An experiment, whose result cannot be uniquely predicted even if the previous results of the same experiment conducted under similar conditions are all known is called a random or more precisely a non-deterministic experiment.

A non-deterministic experiment is also sometimes known as a *trial*. Note that one or more trials may constitute an experiment. A possible result of a random experiment is called its *outcome* or *sample point*.

Tossing of a coin is a non-deterministic experiment. In tossing a coin 10000 times, we may get 5109 heads and 4891 tails. With this prior information, we cannot the predict the outcome of the 10001^{th} toss beforehand.

Definition 14.2 (Sample space and event). The totality of all the possible outcomes of an experiment is called the sample space of the experiment. Any component of a sample space is an event.

In tossing a coin, the possible outcomes are a head (denoted by H) and a tail (denoted by T). If S denotes the collection of these two outcomes, i.e., $S = \{H, T\}$, then S is the sample space. We write $\{H\}$ to denote the event of getting a head, $\{T\}$ to denote the event of getting a tail and $\{H, T\}$ to denote the event of getting a head or a tail. The sample space of tossing two coins once is $\{(H, H), (H, T), (T, H), (T, T)\}$, where $\{(H, T)\}$ is the event of getting a head in the first coin and a tail in the second coin, etc. But for the sake of simplicity, we usually write this sample space as $\{HH, HT, TH, TT\}$. Here, $\{HH\}$ is the event of getting head in both the tosses, $\{HT, TH\}$ is the event of getting only one head (or one tail), etc. Note that the event $\{HT\}$ is different from the event $\{TH\}$.

Example 1. Write down the sample space of tossing a coin and rolling a die simultaneously.

Answer: The sample space of tossing a coin and rolling a die simultaneously is $\{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}$.

Remark: Let us consider the experiment of tossing a coin and rolling a die simultaneously. The coin may show head (H) or tail (T). When the coin shows H, the die may show any of the six numbers 1, 2, 3, 4, 5 and 6. Same is the case when the coin shows T. So, there are $2 \times 6 = 12$ possible outcomes.

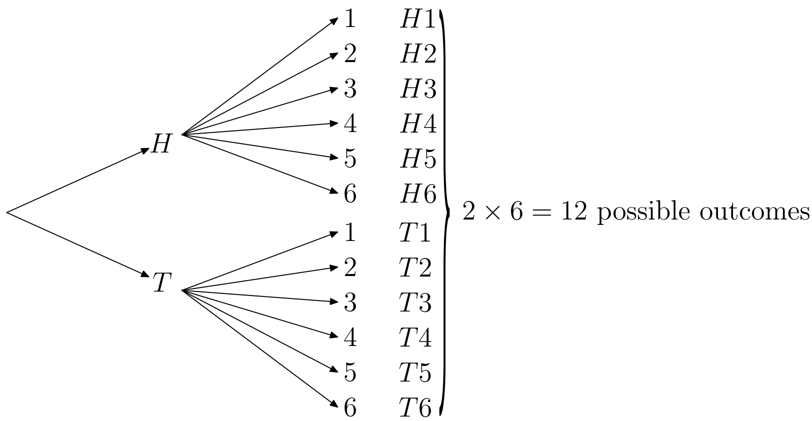


Fig: 14.1 Tree diagram for tossing a coin and rolling a die simultaneously.

Example 2. Write down the sample space of choosing two balls from an urn containing 2 red balls and 3 blue balls.

Answer: If R_1, R_2 denote the red balls and B_1, B_2, B_3 denote the blue balls, then the sample space is

$$\{R_1R_2, R_1B_1, R_1B_2, R_1B_3, R_2B_1, R_2B_2, R_2B_3, B_1B_2, B_1B_3, B_2B_3\}.$$

The order is not important, i.e., R_1R_2 is the same as R_2R_1 and so on.

Example 3. Write down the sample space of choosing two balls one after another from an urn containing 2 red balls and 3 blue balls.

Answer: If R_1, R_2 denote the red balls and B_1, B_2, B_3 denote the blue balls, then the sample space is

$$\{R_1R_2, R_1B_1, R_1B_2, R_1B_3, R_2R_1, R_2B_1, R_2B_2, R_2B_3, B_1R_1, B_1R_2,$$
$$B_1B_2, B_1B_3, B_2R_1, B_2R_2, B_2B_1, B_2B_3, B_3R_1, B_3R_2, B_3B_1, B_3B_2\}.$$

The order is important, i.e., R_1R_2 and R_2R_1 are different outcomes, etc.

Definition 14.3 (Equally likely events). Events are said to be equally likely if there is no valid reason to say that one event has more chance to occur than the others.

In tossing an unbiased coin, there is no valid reason to say that a head (or a tail) has more chance to turn up than the other. The events $\{H\}$ and $\{T\}$ are equally likely.

Exercise 4. Which of the following experiments have equally likely outcomes? Explain.
(a) A boy attempts to win the Fields Medal. He wins or does not win.
(b) A girl participates in an examination. She fails or passes.
(c) A baby is born. It is a boy or a girl.

Hint: (a) No. **(b)** No. **(c)** Yes.

Definition 14.4 (Mutually exclusive events). Events are said to be mutually exclusive if the happening of one prevents the happening of all the others.

In tossing a coin, one of the two faces will turn up. If H turns up, T will not turn up and vice-versa. $\{H\}$ and $\{T\}$ are mutually exclusive events.

Definition 14.5 (Independent events). Two events are said to be independent if the occurrence of one has no effect on the occurrence of the other.

If a coin is tossed twice, the event of getting a head in the first toss and the event of getting a tail in the second toss are independent. Here, the sample space is $\{HH, HT, TH, TT\}$, the event of getting a head in the first toss is $\{HH, HT\}$ and the event of getting a tail in the second toss is $\{HT, TT\}$.

Example 5. Are the two events $\{H\}$ and $\{T\}$ in tossing a coin independent? Justify your answer.

Answer: The two events $\{H\}$ and $\{T\}$ in tossing a coin are not independent. In tossing a coin, the happening head prevents the happening of tail, i.e., if H turns up T will not turn up and vice-versa. In other words, the occurrence of head has affected the occurrence of tail in tossing a coin, etc. Hence, $\{H\}$ and $\{T\}$ are not independent.

Definition 14.6 (**Elementary and compound events**)**.** An event having only one outcome of the experiment is called an elementary (or simple or atomic) event. If an event has more than one outcome of the experiment, it is called a compound event.

The elementary events of a sample space are always mutually exclusive. The sample space of tossing a coin twice is $\{HH, HT, TH, TT\}$. There are four elementary events corresponding to this sample space. They are $\{HH\}$, $\{HT\}$, $\{TH\}$ and $\{TT\}$. The event $\{HT, TT\}$ is a compound event formed by two elementary events $\{HT\}$ and $\{TT\}$.

Definition 14.7 (**Exhaustive events**)**.** A set of events is said to be exhaustive if all the possible outcomes are included.

In tossing a coin once, the two events {H} and {T} constitute a set of exhaustive events.

Definition 14.8 (**Favourable outcomes**)**.** Out of the set of exhaustive outcomes, those entailing the occurrence of a particular event are called the favourable ones for the event.

In tossing a coin twice, out of the four exhaustive outcomes, only three are favourable to the event of getting at least one H.

Definition 14.9 (**Classical or mathematical or a priori definition of probability due to Laplace**)**.** Out of n exhaustive, equally likely and mutually exclusive outcomes, if m are favourable to the event A, then the classical (or theoretical) probability of the occurrence of the event A denoted by $P(A)$ is the ratio $m : n$, i.e.,

$$P(A) = \frac{\text{number of outcomes favourable to } A}{\text{total number of possible outcomes}} = \frac{m}{n}.$$

Exercise 6. $0 \leq P(A) \leq 1$ for any event A. Justify.

Hint: Out of n exhaustive, equally likely, mutually exclusive outcomes, if m are favourable to A, then $0 \leq m \leq n$.

Example 7. What is the probability that a number selected at random from the first 10 natural numbers is a prime number?

Solution: If S is the sample space and E the event of selecting a prime number, then

$$S = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, \quad E = \{2, 3, 5, 7\}.$$

$$\therefore\ P(E) = \frac{\mathrm{n}(E)}{\mathrm{n}(S)} = \frac{\text{number of elements in } E}{\text{number of elements in } S} = \frac{4}{10} = \frac{2}{5}.$$

Definition 14.10 (Sure and impossible events). An event of probability 1 is called a sure (or certain) event and an event of probability 0 is called an impossible event.

In tossing a die, the event of getting a number less than 10 is a sure event whereas the event of getting 10 is an impossible event.

Definition 14.11 (Complementary events). Two events are said to be complementary if the sum of their probabilities is 1. The complementary event of an event A is denoted by $\overline{A}$ or A^c.

In tossing a die, the event of getting a number not greater than 4 and the event of getting a number greater than 4 are complementary.

Exercise 8. If A is any event, then $P(A) + P(\overline{A}) = 1$, where $\overline{A}$ is the complement of A. Justify.

Hint: Out of n exhaustive, equally likely, mutually exclusive outcomes, if m are favourable to A, the remaining $n - m$ outcomes are not favourable to the event A.

Notes:

1. If there are n outcomes of a random experiment, then there are 2^n possible events.
2. When we say "independent events," we are referring to events with the same sample space. The same is the case with "equally likely events", "mutually exclusive events", "exhaustive events", etc.
3. In classical probability, we assume that all outcomes of a random experiment are equally likely.

4. The probability of an event may be an irrational number. See example 16 on page 481.
5. The sum of the probabilities of all the elementary events of an experiment is 1.
6. The sum of the probabilities of mutually exclusive events forming an exhaustive system is 1. In particular, if A, B and C are mutually exclusive and exhaustive events, then $P(A) + P(B) + P(C) = 1$.
7. If A and B are two events, then $A \cup B$ denotes the event of happening at least one of A and B, and AB (also denoted by $A \cap B$) denotes the event of the combined occurrence of A and B.
8. Two events A and B are said to be independent if and only if $P(AB) = P(A) \cdot P(B)$. Otherwise, A and B are called dependent events.
9. Mutually exclusive events have no common outcome.
10. Two independent events always have at least one common outcome if they are not impossible events.
11. (**Addition theorem**) If A and B are mutually exclusive events, then
$$P(A \cup B) = P(A) + P(B).$$
In general, if A and B are any two events, then
$$P(A \cup B) = P(A) + P(B) - P(AB).$$

Exercise 9. Why is tossing a coin considered to be a fair way of deciding which team should get the ball at the beginning of a football match?

Exercise 10. Two coins are tossed once. What is the probability of getting three heads? (Answer: 0.)

Exercise 11. Two fair dice are rolled. What is the probability that the sum of the points is at least 2? (Answer: 1.)

Exercise 12. In two independent tosses of a fair die the sum of the outcomes was 9. What would be the probability that the first toss resulted in 6? (Answer: 1/4.)

Exercise 13. A bag contains 30 balls out of which some are red, some are blue and remaining are black. If the probability of drawing a red ball is 11/15 and that of a blue ball is 1/10, then how many black balls are there in the bag. (Answer: 5.)

Example 14. Khamba and Thoibi are friends. Find the probability that **(i)** they have different birthdays; **(ii)** they have the same birthday. Assume that they were not born on a leap year.

Solution: Khamba's birthday can be any day of the 365 days of the year. Also, Thoibi's birthday can be any day of the year. We assume that these 365 outcomes are equally likely.

(i) If Khamba's birthday is different from Thoibi's birthday, the number of favourable outcomes of Thoibi's birthday $= 365 - 1 = 364$.

Let E be the event that they have different birthdays. Then the probability that they have different birthdays, $P(E) = \dfrac{364}{365}$.

(ii) The event that they have the same birthday and the event that they have different birthdays are complementary. Hence, the probability that they have the same birthday $= P(\overline{E}) = 1 - P(E) = 1 - \dfrac{364}{365} = \dfrac{1}{365}$.

Do you know? In a group of people, what is the probability that two of them have the same birthday? This is the birthday problem or birthday paradox. The probability is 100% when the number of people is more than 366 as there are only 366 possible birthdays. However, the probability reaches 99.9% with just 70 people and 50% with just 23 people. There is a cryptographic attack called the birthday attack that exploits the mathematics behind the birthday problem. Note that the birthday problem is different from finding the probability that, in a group of people, someone has the same birthday as you (or a particular person). In order to get more that 50% probability in this case, the number of people must be at least 253.

Example 15. If a leap year is selected at random, what is the probability that it will have 53 Sundays?

Solution: We know that a leap year has 366 days, i.e., 52 weeks and 2 days. These two days will be two consecutive days of a week. The sample space for the possible pairs of days is

$$\begin{aligned} S = \{&(\text{Sunday}, \text{Monday}), (\text{Monday}, \text{Tuesday}), (\text{Tuesday}, \text{Wednesday}),\\ &(\text{Wednesday}, \text{Thursday}), (\text{Thursday}, \text{Friday}), (\text{Friday}, \text{Saturday}),\\ &(\text{Saturday}, \text{Sunday})\}.\end{aligned}$$

Here, (Sunday, Monday) and (Saturday, Sunday) are the favourable cases. Therefore, the required probability $= \dfrac{\text{number of favourable cases}}{\text{total number of cases}} = \dfrac{2}{7}$.

Remark: In the Gregorian calendar, those years exactly divisible by 100, but not by 400, are not leap years. For example, the years 1700, 1800, and 1900 are not leap years, but the year 2000 is a leap year. A leap year having 53 Sundays must start on a Saturday or a Sunday. The distribution of days of the week repeats exactly every 400 years. Within these 400 years, there are 97 leap years, out of which 13 start on a Saturday and 15 start on a Sunday. So, the probability that a leap year chosen at random has 53 Sundays is 28/97 rather than 2/7 = 28/98.

Example 16. A circular region of radius 7 m is inside a rectangular field of dimension 70 m $\times$ 21 m. A cow is grazing in the field. Find the probability that the cow is grazing in the circular region.

Solution: The cow is equally likely to graze anywhere in the field. The area of the entire field where the cow can graze $= 70 \times 21$ m^2. Also, the area of the circular region $= \pi \times 7^2 = 49\pi$ m^2.

The probability that the cow is grazing in the circular region

$$= \frac{\text{area of the circular region}}{\text{area of the rectangular field}} = \frac{49\pi}{70 \times 21} = \frac{\pi}{30}.$$

A note on playing cards: There are 52 cards in a pack (or deck) of cards which are divided into 4 suits of 13 cards each. The cards in each suit are Ace, King, Queen, Jack (or Knave), 10, 9, 8, 7, 6, 5, 4, 3 and 2. King, Queen and Jack are the face cards. Ace, King, Queen and Jack are the power cards. The colours and number of cards in each of the four suits in a pack are given below:

Suit name	Colour	Number of Cards			
		Ace	Face Cards	Numeral Cards	Total
Heart ($\heartsuit$)	Red	1	3	9	13
Diamond ($\diamondsuit$)	Red	1	3	9	13
Club ($\clubsuit$)	Black	1	3	9	13
Spade ($\spadesuit$)	Black	1	3	9	13

Exercise 17. What is the probability of drawing a Queen from a pack of well-shuffled cards. (Answer: 1/13.)

Example 18. If a coin is tossed twice, the event of getting a head in the first toss and the event of getting a tail in the second toss are independent. Prove it.

Solution: The sample space is $\{HH, HT, TH, TT\}$. Let A be the event of getting a head in the first toss and B be the event of getting a tail in the second toss. Then $A = \{HH, HT\}$ and $B = \{HT, TT\}$. So, $AB = \{HT\}$. Now,

$$P(A) = \frac{2}{4} = \frac{1}{2},\ P(B) = \frac{2}{4} = \frac{1}{2} \text{ and } P(AB) = \frac{1}{4}.$$

Clearly, we see that $P(A) \cdot P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(AB)$. This shows that A and B are independent. $\square$

Exercise 19. Give one example of each of the following:
(a) Two events which are mutually exclusive but not independent.
(b) Two events which are independent but not mutually exclusive.
(c) Two events which are neither independent nor mutually exclusive.
(d) Two events which are both independent and mutually exclusive.

Independent Events and Independent Experiments

Two experiments are said to be *independent* if the outcome of one experiment is not influenced in any way by the outcome of the other experiment; otherwise they are said to be *interdependent*. The experiment of tossing a coin and the experiment of rolling a die are independent experiments. Now, consider two experiments – (i) drawing a card from a pack of cards, (ii) drawing another card from the remaining cards. Here, the outcome of the second experiment depends on the outcome of the first experiment. Such experiments are said to be interdependent.

We now discuss the probability of an event in terms of the probabilities of events in a sequence of experiments.

(a) Let us consider two independent experiments – (i) tossing a coin and (ii) rolling a die.

Let A_1 be the event of getting a head in the first experiment. Here, the sample space is $\{H, T\}$ and $A_1 = \{H\}$. Clearly, $P(A_1) = \frac{1}{2}$.

Let B_2 be the event of getting 5 in the second experiment. Here, the sample space is $\{1, 2, 3, 4, 5, 6\}$ and $B_2 = \{5\}$. Clearly, $P(B_2) = \frac{1}{6}$.

A *single composite experiment* may be defined to consist of first performing an experiment and then another experiment. Now, let us consider the composite experiment of first tossing a coin and then rolling a die. This experiment can be considered the same as the experiment of tossing a coin and rolling a die simultaneously (because the two experiments are independent). The sample space S is given by

$$S = \{H1, H2, H3, H4, H5, H6, T1, T2, T3, T4, T5, T6\}.$$

Let A be the event of getting head in the coin (with any outcome in the die) and B be the event of getting 5 in the die (with any outcome in the coin).

We have $A = \{H1, H2, H3, H4, H5, H6\}$ and $B = \{H5, T5\}$. So, it is easy to see that $P(A) = \dfrac{6}{12} = \dfrac{1}{2}$ and $P(B) = \dfrac{2}{12} = \dfrac{1}{6}$. Note that $P(A) = P(A_1)$ and $P(B) = P(B_2)$.

The event AB is the simultaneous occurrence of the events A and B. It is the event of getting head in the coin and 5 in the die. We have $AB = \{H5\}$. Therefore, $P(AB) = \dfrac{1}{12}$.

Now, $P(A) \cdot P(B) = \dfrac{1}{2} \times \dfrac{1}{6} = \dfrac{1}{12} = P(AB)$. This shows that A and B are independent events.

From the above discussion, we see that

$$P(A) = P(A_1),\ P(B) = P(B_2),\ P(AB) = P(A) \cdot P(B) = P(A_1) \cdot P(B_2).$$

We generalise the above observations and state the following result. The proof of the result is beyond the scope of this book.

> Let us consider two independent experiments. Let A_1 be an event in the first experiment. Let A be the event in the composite experiment (involving the two independent experiments) in which A_1 occurs in the first experiment together with any outcome in the second experiment. Let B_2 be an event in the second experiment. Let B be the event in the composite experiment in which B_2 occurs in the second experiment together with any outcome in the first experiment. The event AB is the concurrence of the event A and event B. In other words, AB is the event in the composite experiment in which A_1 occurs in the first experiment and B_2 occurs in the second experiment.
>
> Then the following relations are true.
>
> **(1)** $P(A) = P(A_1)$.
> **(2)** $P(B) = P(B_2)$.
> **(3)** $P(AB) = P(A) \cdot P(B)$.
> **(4)** $P(AB) = P(A_1) \cdot P(B_2)$.

The relation **(3)** means that A and B are independent events in the composite experiment. Note that the events A and A_1 are different events defined on different sample spaces; so are the events B and B_2.

(b) Let us consider an experiment of choosing a ball from an urn containing 2 white and 3 black balls. There are 2 white balls out of the 5 balls. If p_1 is the probability of choosing a white ball, then $p_1 = \frac{2}{5}$.

Again, let us consider a second experiment of choosing another ball from the urn after the first experiment has been performed (and without replacing the ball chosen in the first experiment). Now, there are 4 balls in the urn but the number of white or black balls depends on the outcome of the first experiment. This shows that the experiments are interdependent. Suppose a white ball has been chosen in the first experiment. Then there are 1 white and 3 black balls after the first experiment has been performed. If p_2 is the probability of choosing a black ball in the second experiment after a white ball has been chosen in the first experiment, then $p_2 = \frac{3}{4}$.

Now, let us consider an experiment of choosing two balls one after another from an urn containing 2 white ball and 3 black balls. The sample space is

$$\{W_1W_2, W_1B_1, W_1B_2, W_1B_3, W_2W_1, W_2B_1, W_2B_2, W_2B_3, B_1W_1, B_1W_2,\\ B_1B_2, B_1B_3, B_2W_1, B_2W_2, B_2B_1, B_2B_3, B_3W_1, B_3W_2, B_3B_1, B_3B_2\},$$

where W_1, W_2 denote the white balls and B_1, B_2, B_3 denote the black balls.

Let A be the event that the first ball chosen is white (i.e., first a white ball and then a ball of any colour are chosen). Let B be the event that the second ball chosen is black (i.e., first a ball of any colour and then a black ball are chosen).
Then $A = \{W_1W_2, W_1B_1, W_1B_2, W_1B_3, W_2W_1, W_2B_1, W_2B_2, W_2B_3\}$ and $B = \{W_1B_1, W_1B_2, W_1B_3, W_2B_1, W_2B_2, W_2B_3, B_1B_2, B_1B_3, B_2B_1, B_2B_3, B_3B_1, B_3B_2\}$. Clearly, $P(A) = \frac{8}{20} = \frac{2}{5}$ and $P(B) = \frac{12}{20} = \frac{3}{5}$.

The event AB is the concurrence of the events A and B. It is the event of choosing first a white ball and then a black ball. Now, $P(AB) = \frac{6}{20} = \frac{3}{10}$ because $AB = \{W_1B_1, W_1B_2, W_1B_3, W_2B_1, W_2B_2, W_2B_3\}$.

Let $B|A$ denote the happening of B on the supposition that A has already happened. Note that $B|A$ is not an event. However, $B|A$ may be considered as the event $\{W_1B_1, W_1B_2, W_1B_3, W_2B_1, W_2B_2, W_2B_3\}$ (i.e., the collection of those outcomes in A in which the 2nd ball chosen is black) defined on A as the sample space. So, $P(B|A) = \frac{6}{8} = \frac{3}{4}$.

$P(B|A)$ is called the *conditional probability of* B *given* A. You will learn more about it in higher classes.

Note that $P(A) = p_1$, $P(B|A) = p_2$, but $P(B) \neq p_2$.
Also, $P(AB) = P(A) \cdot P(B|A) = p_1 p_2$.

Here, $P(AB) \neq P(A) \cdot P(B)$, i.e., A and B are not independent events.

We generalise the above observations and state the following result. The proof of the result is beyond the scope of this book.

> Let us consider a sequence of experiments. If p_1 is the probability of an event E_1 in the first experiment and p_2 is the probability of an event E_2 in the second experiment *on the supposition that E_1 has happened*. Then the probability that both E_1 and E_2 will happen is $p_1 p_2$.
>
> Also, if p_3 is the probability of an event E_3 in a third experiment *after E_1 and E_2 have happened*, then the probability that all of E_1, E_2 and E_3 will happen is $p_1 p_2 p_3$ and so on for any number of events.

The above result holds true even when the experiments are independent.

Example 20. Two coins are tossed simultaneously. Find the probability of getting a head in the first coin and a tail in the second coin.

Solution: Let A be the event of getting a head in the first coin. Let B be the event of getting a tail in the second coin. Clearly, we have $P(A) = \frac{1}{2}$, $P(B) = \frac{1}{2}$. The happening of A does not affect the happening of B and vice-versa. The events A and B are independent. So, the probability of getting a head in the first coin and a tail in the second coin, $P(AB) = P(A)P(B) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Example 21. Two balls are chosen from an urn containing two red balls and three blue balls. Find the probability that both balls are blue.

Solution: The urn contains 2 red balls and 3 blue balls. Out of the 5 balls a blue ball can be chosen in 3 ways. So, the probability of choosing a blue ball $= \frac{3}{5}$.

Assuming a blue ball has already been chosen, the urn now contains 2 red balls and 2 blue balls. Out of the 4 balls a blue ball can be chosen in 2 ways. So, the probability that the second ball chosen is blue $= \frac{2}{4} = \frac{1}{2}$.
Hence, the probability that both the balls chosen are blue $= \frac{3}{5} \times \frac{1}{2} = \frac{3}{10}$.

Exercise 22. Two balls are chosen at random from an urn containing two red balls and three blue balls. Find the probability that the balls are of different colours.

Hint: Different balls can be chosen in two different ways – (i) first a red ball and then a blue ball (ii) first a blue ball and then a red ball. If A is the event of choosing first a red ball and then a blue ball, then $P(A) = \frac{2}{5} \times \frac{3}{4} = \frac{3}{10}$. If B is the event of choosing first a blue ball and then a red ball, then $P(B) = \frac{3}{5} \times \frac{2}{4} = \frac{3}{10}$. Note that $P(A \cup B)$ is the probability of the happening of A or B. In the present case, it is the probability that the two balls chosen are of different colours. Since A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B) = \frac{3}{10} + \frac{3}{10} = \frac{6}{10} = \frac{3}{5}$.

Exercise 23. Two balls are drawn from an urn containing 3 red balls and 5 black balls. Find the probability that at least one ball is red.

Hint: The probability that both the balls drawn is black is $\frac{5}{8} \times \frac{4}{7} = \frac{5}{14}$. So, the probability that at least one ball is red $= 1 - \frac{5}{14} = \frac{9}{14}$.

Exercise 24. Two cards are drawn from a pack of cards. Find the probability that the first card is a King and the second card is a Queen.

Hint: The probability is $\frac{4}{52} \times \frac{4}{51} = \frac{4}{663}$.

Exercise 25. From a pack of cards, two cards are drawn at random (one after another without replacing). Find the probability that the first one is a red face card and the second is a black face card. (Answer: 3/221.)

Exercise 26. A pack of card is dealt out.
(a) What is the probability that the fifth card dealt is a King?
(b) What is the probability that the first King occurs on the fifth card?

Hint: **(a)** The probability is $\frac{4}{52} = \frac{1}{13}$ because we are not given any information on what has happened in the previous deals. **(b)** None of the first 4 cards dealt is a King and the 5th card dealt is a King. So, the probability $= \left(1 - \frac{1}{13}\right) \cdot \left(1 - \frac{1}{13}\right) \cdot \left(1 - \frac{1}{13}\right) \cdot \left(1 - \frac{1}{13}\right) \cdot \left(\frac{1}{13}\right) = \frac{12^4}{13^5}$.

Exercise 27. A fair coin is tossed repeatedly. What is the probability of getting a head in the third toss and a tail in the fourth toss? (Answer: 1/4.)

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