Higher Mathematics X

Based on the syllabus prescribed by the BSEM

For Class X

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Front Cover: Cartesian plane showing the signs of the trigonometric ratios in each quadrant.

Back Cover: Pascal's triangle up to 6th row.

HIGHER MATHEMATICS X

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To all the brave, hardworking and lovely mothers and fathers who would face immense hardships to educate their children

Preface

We have written this book bearing in mind the needs of those students of class X who opt Higher Mathematics. This book is meant to help the students get all the materials they need without having to rely on other books. It is made in a compact and comprehensive form and it includes all the details a student needs to understand the subject matters properly. In order to make the readers familiarise themselves with the question pattern, we have added some question papers of the HSLC Examination along with their solutions.

We would like to thank those who gave suggestions, advice and support. In particular, we wish to thank Hidam John Angom for his various comments and suggestions.

We welcome feedbacks and suggestions from our readers for further improvement of this book.

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Notations

$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}$	the set of integers
$\mathbb{Q}$	the set of rational numbers
$\mathbb{R}$	the set of real numbers

$\Longleftrightarrow$	if and only if
$\Longrightarrow$	implies
$\forall$	for all
$\exists$	there exists
$x \in A$	the element x belongs to the set A
$A \cup B$	the union of A and B
$A \cap B$	the intersection of A and B
$n!$	the product of first n natural numbers
(x, y)	the coordinates of a point
$\Box$	Q.E.D. (quod erat demonstrandum), that which was to be demonstrated

Contents

Chapter 1

Binary Operations

> *"But that's – I'm sorry, but that's completely ridiculous! How can I* possibly *prove it doesn't exist? Do you expect me to get hold of – of all the pebbles in the world and test them? I mean, you could claim that* anything's *real if the only basis for believing in it is that nobody's* proved *it doesn't exist!"*
>
> — J. K. Rowling, *Harry Potter and the Deathly Hallows*

A binary operation on a set is a rule that combines two elements of the set to produce a new element of the set. The usual addition, subtraction, multiplication and division are the most commonly known binary operations.

Binary Operation on a Set

Definition 1.1 (Binary operation). Let S be a non-empty set and $\circ$ be a mapping of the cartesian product $S \times S$ to S. Then $\circ$ is called a binary operation on the set S.

Thus, a binary operation $\circ$ on a set S assigns to each ordered pair $(x, y) \in S \times S$ a uniquely determined element $x \circ y \in S$. The element $x \circ y$ is the $\circ$-image of the pair (x, y); it is called the composite (or product) of x and y under $\circ$.

The usual addition of natural numbers is a binary operation on $\mathbb{N}$. For each ordered pair $(x, y) \in \mathbb{N} \times \mathbb{N}$, the element $x + y \in \mathbb{N}$ is uniquely defined.

If $\circ$ is a binary operation on a set S and $a, b \in S$ are any two elements, then $a \circ b \in S$, by definition. Sometimes we express this fact by saying that S is closed under $\circ$.

Definition 1.2 (**External binary operation**). Let A and S be non-empty sets and $f : A \times S \to S$ be a mapping. Then f is called an external binary operation on S over A.

Thus, an external binary operation f on S over A assigns to each ordered pair $(a, x) \in A \times S$ a uniquely defined element $f(a, x) \in S$. The scalar multiplication of vectors is an example of an external binary operation on the set of vectors over the set of scalars.

In the remainder of this chapter, we will restrict our discussion to binary operation as defined in definition 1.1 only.

Definition 1.3 (**Algebraic structure**). A set equipped with one or more binary operations is called an algebraic structure.

If $\circ$ is a binary operation on a set S, then the pair $(S, \circ)$ is an algebraic structure. The set $\mathbb{R}$ of real numbers equipped with the usual addition $+$ and the usual multiplication $\times$ forms an algebraic structure called the field of real numbers. It is denoted by $(\mathbb{R}, +, \times)$.

Definition 1.4. A binary operation $\circ$ on a set S is said to be commutative if $x \circ y = y \circ x$ for all $x, y \in S$.

Definition 1.5. A binary operation $\circ$ on a set S is said to be associative if $(x \circ y) \circ z = x \circ (y \circ z)$ for all $x, y, z \in S$.

For a binary operation $\circ$ which is not associative, the expression $x_1 \circ x_2 \circ \cdots \circ x_n$ is ambiguous unless brackets are used. However, if $\circ$ is associative, the expression $x_1 \circ x_2 \circ \cdots \circ x_n$ is uniquely defined; the factors may be grouped in any manner so long as the order of the elements is unchanged. Besides being associative, if $\circ$ is commutative also, then the order of the factors may also be changed randomly without altering the value of the product.

Definition 1.6. Let $\circ$ be an associative binary operation on a set S. Then for any $x \in S$ and any $n \in \mathbb{N}$, the nth power of x, denoted by x^n, is defined by $x^n = x \circ \cdots \circ x$ (n factors, each equals to x).

It is easy to prove that $x^m \circ x^n = x^{m+n}$ for any $m, n \in \mathbb{N}$.

In case an associative binary operation $+$ is denoted additively, the nth multiple (additive power) of x, denoted by nx, is defined by $nx = x + \cdots + x$ (n terms, each equals to x). Also, $mx + nx = (m + n)x$ for any $m, n \in \mathbb{N}$.

Definition 1.7. Let $*$ and $\circ$ be two binary operations on a set S. Then $*$ is said to be distributive over $\circ$ if $x * (y \circ z) = (x * y) \circ (x * z)$ and $(y \circ z) * x = (y * x) \circ (z * x)$ for all $x, y, z \in S$.

Definition 1.8. Let $\circ$ be a binary operation on a set S and let $H \subseteq S$. Then H is said to be closed under $\circ$ if $a \circ b \in H$ for any $a, b \in H$.

For the algebraic structure $(S, \circ)$, if H is a subset of S closed under $\circ$, then $\circ$ is a binary operation on H also, i.e., $(H, \circ)$ is also an algebraic structure.

Example 1. Prove that the binary operation $*$ on the set $\mathbb{N} \cup \{0\}$ defined by $a * b = (a - b)^2$, $\forall\ a, b \in \mathbb{N} \cup \{0\}$, is commutative but not associative.

Solution: For any $a, b \in \mathbb{N} \cup \{0\}$, we have

$$a * b = (a - b)^2 = (b - a)^2 = b * a.$$

Hence, the binary operation $*$ is commutative on $\mathbb{N} \cup \{0\}$.

Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * ((2 - 3)^2) = 1 * 1 = (1 - 1)^2 = 0,$$
$$(a * b) * c = (1 * 2) * 3 = ((1 - 2)^2) * 3 = 1 * 3 = (1 - 3)^2 = 4.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{N} \cup \{0\}$. Hence, the operation $*$ is not associative. $\square$

Remark: In order to prove a statement, we need a logical argument. However, to disprove a statement, a single counterexample is enough.

Composition Table

If S is a finite set, consisting of n elements say, then a binary operation $\circ$ in S can be described by means of a table consisting of n rows and n columns in which the entry at the intersection of the row headed by an element $a \in S$ and the column headed by an element $b \in S$ is $a \circ b$. Such a table is called a composition table.

Example 2. Consider the set $S = \{2, 3, 4, 5, 6\}$ with the binary operation $*$ defined by

$$a * b = \text{the greatest prime factor of } ab.$$

Form the composition table of $(S, *)$.

Solution: Put $*$ and the elements of S in the topmost row and the leftmost column (see the table below). Then we find out the values of $a * b$, where $a, b \in S$. When $a = 2$ and $b = 2$, we have

$$a * b = 2 * 2 = (\text{the greatest prime factor of } 2 \times 2) = 2.$$

Put this value in the entry which is at the intersection of the row headed by $a = 2$ and the column headed by $b = 2$. Similarly, we can find out the values of the remaining entries. Now, we form the composition table for $(S, *)$ as follows.

$*$	2	3	4	5	6
2	2	3	2	5	3
3	3	3	3	5	3
4	2	3	2	5	3
5	5	5	5	5	5
6	3	3	3	5	3

Definition 1.9. An algebraic structure $(S, \circ)$ is said to be with an identity element e if there exists $e \in S$ such that $x \circ e = e \circ x = x$ for every $x \in S$.

Theorem 1.1. *The identity element for an algebraic structure, if it exists, is unique.*

Proof: Let $(S, \circ)$ be an algebraic structure. If e_1 and e_2 are two identity elements of $(S, \circ)$, then

$$\begin{aligned} e_1 \circ e_2 &= e_2 \quad (\because e_1 \text{ is an identity element}), \\ e_1 \circ e_2 &= e_1 \quad (\because e_2 \text{ is an identity element}). \end{aligned}$$

But $e_1 \circ e_2$ is uniquely determined because $e_1 \circ e_2$ is the composition of e_1 and e_2. So, $e_1 = e_2$. Hence, the identity element of an algebraic structure, if it exists, is unique. □

Definition 1.10. Let $(S, \circ)$ be an algebraic structure with identity element e and let x be an element of S. An element $y \in S$, if it exists, is said to be an inverse of x if $x \circ y = y \circ x = e$. The inverse of x is usually denoted by x^{-1}.

Theorem 1.2. *If $(S, \circ)$ is an algebraic structure with identity in which the binary operation $\circ$ is associative, then the inverse of an element of S, if it exists, is unique.*

Proof: If y and z are two inverse elements of an element $x \in S$, then

$$x \circ y = y \circ x = e, \tag{1}$$

$$x \circ z = z \circ x = e, \text{ where } e \text{ is the identity element.} \tag{2}$$

Now, we have

$$\begin{aligned} y &= y \circ e && (\because e \text{ is the identity element}) \\ &= y \circ (x \circ z), && \text{by (2)} \end{aligned}$$

$$\begin{aligned} &= (y \circ x) \circ z && (\because \circ \text{ is associative}) \\ &= e \circ z, && \text{by (1)} \\ &= z && (\because e \text{ is the identity element}). \end{aligned}$$

Hence, the inverse of an element of S, if it exists, is unique. □

Definition 1.11. In an algebraic structure with identity, an element is said to be invertible if its inverse exists.

Theorem 1.3. *Let $(S, \circ)$ be an algebraic structure with identity in which the binary operation $\circ$ is associative. If x and y are two invertible elements of S, then $x \circ y$ is also invertible and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$.*

Proof: Let e be the identity element of $(S, \circ)$. Since x^{-1} and y^{-1} are the inverse elements of x and y respectively, we have

$$x \circ x^{-1} = x^{-1} \circ x = e, \tag{1}$$

$$y \circ y^{-1} = y^{-1} \circ y = e. \tag{2}$$

Now, we have

$$\begin{aligned} (x \circ y) \circ (y^{-1} \circ x^{-1}) &= \left[(x \circ y) \circ y^{-1}\right] \circ x^{-1} && (\because \circ \text{ is associative}) \\ &= \left[x \circ (y \circ y^{-1})\right] \circ x^{-1} && (\because \circ \text{ is associative}) \\ &= (x \circ e) \circ x^{-1}, && \text{by (2)} \\ &= x \circ x^{-1} && (\because e \text{ is the identity element}) \\ &= e, && \text{by (1)}. \end{aligned}$$

Similarly, we have

$$\begin{aligned} (y^{-1} \circ x^{-1}) \circ (x \circ y) &= \left[(y^{-1} \circ x^{-1}) \circ x\right] \circ y && (\because \circ \text{ is associative}) \\ &= \left[y^{-1} \circ (x^{-1} \circ x)\right] \circ y && (\because \circ \text{ is associative}) \\ &= (y^{-1} \circ e) \circ y, && \text{by (1)} \\ &= y^{-1} \circ y && (\because e \text{ is the identity element}) \\ &= e, && \text{by (2)}. \end{aligned}$$

Thus, we see that

$$(x \circ y) \circ (y^{-1} \circ x^{-1}) = e = (y^{-1} \circ x^{-1}) \circ (x \circ y).$$

And, since $x^{-1}, y^{-1} \in S$ and S is closed under $\circ$, the element $y^{-1} \circ x^{-1} \in S$. Hence, $x \circ y$ is invertible and $(x \circ y)^{-1} = y^{-1} \circ x^{-1}$. □

Example 3. Let n be a fixed positive integer and let $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$. For all $a, b \in \mathbb{Z}_n$, let $a +_n b$ denote the remainder when $a + b$ is divided by n, and let $a \times_n b$ denote the remainder when ab is divided by n. Then

$+_n$ and $\times_n$ are both binary operations in $\mathbb{Z}_n$. The set $\mathbb{Z}_n$ is usually called the **set of integers modulo** n. The operation $+_n$ is usually called **addition modulo** n, and the operation $\times_n$ is usually called **multiplication modulo** n.

(a) Show that both $+_n$ and $\times_n$ are commutative as well as associative.

(b) Let $\circ$ be a binary operation on $\mathbb{Z}_7$ defined by $a \circ b = (a +_7 b) \times_7 b$, for all $a, b \in \mathbb{Z}_7$. Check, by means of a composition table, whether the set $H = \{0, 1, 2, 3\} \subset \mathbb{Z}_7$ is closed under $\circ$ or not.

(c) Show that the binary operation $\circ$ defined above (in (b)) is neither commutative nor associative.

(d) Examine the algebraic structures $(\mathbb{Z}_7, +_7)$ and $(\mathbb{Z}_7, \times_7)$ for the existence of identity and invertible elements.

Solution:

(a) For any $a, b \in \mathbb{Z}_n$, we have

$$\begin{aligned} a +_n b &= \text{the remainder when } a + b \text{ is divided by } n \\ &= \text{the remainder when } b + a \text{ is divided by } n \\ &= b +_n a. \end{aligned}$$

Hence, $+_n$ is commutative. The remaining parts are left as an exercise for the reader. □

(b) We form the composition table for $(H, \circ)$ as follows.

$\circ$	0	1	2	3
0	0	1	4	2
1	0	2	6	5
2	0	3	1	1
3	0	4	3	4

From the table we see that $1 \circ 2 = 6 \notin H$, etc. Hence, H is not closed under $\circ$.

(c) Left as an exercise for the reader.

(d) Let us consider the algebraic structure $(\mathbb{Z}_7, +_7)$. If e is the identity element of $(\mathbb{Z}_7, +_7)$, then for any element $a \in \mathbb{Z}_7$, we have

$$a +_7 e = e +_7 a = a.$$

This means that a is the remainder when $a + e$ (or $e + a$) is divided by 7. This is possible only when $e = 0 \in \mathbb{Z}_7$ for any $a \in \mathbb{Z}_7$. Hence, $e = 0$ is the identity element of $(\mathbb{Z}_7, +_7)$.

Now, if $y \in \mathbb{Z}_7$ is the inverse of $x \in \mathbb{Z}_7$, then we have

$$x +_7 y = y +_7 x = 0.$$

This means that $x + y$ (or $y + x$) is divisible by 7. So, we must have $x + y = 0$ or $x + y = 7$. Thus, for $x = 0$, we see that $y = 0$, and for $x \neq 0$, we see that $y = 7 - x$. Hence, the inverse of 0 is 0 and the inverse of a non-zero $x \in \mathbb{Z}_7$ is $7 - x$.

For the algebraic structure $(\mathbb{Z}_7, \times_7)$, the identity element is 1, the element 0 is not invertible and $1^{-1} = 1$, $2^{-1} = 4$, $3^{-1} = 5$, $4^{-1} = 2$, $5^{-1} = 3$, $6^{-1} = 6$. The details are left as an exercise for the reader.

Exercise 4. Consider the binary operation $\circ$ on $\mathbb{N}$ defined by $a \circ b =$ minimum of a and b. Prove that $\circ$ is commutative. Examine the algebraic structure $(\mathbb{N}, \circ)$ for the existence of identity and invertible elements.

Exercise 5. Prove that the binary operation $\circ$ on the set $\mathbb{R}$ defined by

$$a \circ b = \begin{cases} 1 & \text{if } a > b, \\ 0 & \text{if } a = b, \\ -1 & \text{if } a < b \end{cases}$$

is neither commutative nor associative.

Example 6. How many different binary operations can be defined on a set S consisting of 2 elements?

Answer: 2^4.

Explanation: There are $2 \times 2 = 4$ elements in $S \times S$. Each of these 4 elements can map to any of the 2 elements of S. So, there are $2 \times 2 \times 2 \times 2 = 2^4$ possible mappings from $S \times S$ to S. Another approach via composition table: If a composition table of S is constructed, there are $2 \times 2 = 4$ entries to be filled. Each of these 4 entries can be any of the 2 elements of S (so that the composition is a binary operation). So, there are $2 \times 2 \times 2 \times 2 = 2^4$ possible ways of constructing the composition table.

Exercise 7. Show that the binary operation $*$ on $\mathbb{R} \times \mathbb{R}$ defined by $(a, b) * (c, d) = (ac - bd, ad + bc)$ is commutative as well as associative. Also, examine the algebraic structure $(\mathbb{R} \times \mathbb{R}, *)$ for the existence of identity and invertible elements.

Hint. The identity element is $(1, 0)$ and the inverse of (a, b) is

$$\left(\frac{a}{a^2 + b^2}, \frac{-b}{a^2 + b^2} \right),$$

where $(a, b) \neq (0, 0)$.

Exercise 1.1

1. If E is the set of all even natural numbers and F, the set of all odd natural numbers, answer the following:

   (a) Is addition a binary operation on F?

   (b) Is multiplication a binary operation on F? If yes, find whether identity element exists or not.

   (c) Is addition a binary operation on E? If yes, find whether identity element exists or not.

   (d) Is multiplication a binary operation on E? If yes, find whether identity element exists or not.

   Solution:

   (a) No. Addition is not a binary operation on F. We see that $1, 3 \in F$, but $1 + 3 = 4 \notin F$.

   (b) Yes. Multiplication is a binary operation on F. We know that every number in F is of the form $2k - 1$ for some $k \in \mathbb{N}$. Now, $(2a - 1)(2b - 1) = 2(2ab - a - b + 1) - 1 \in F$ for any $a, b \in \mathbb{N}$ (because $2ab - a - b + 1 = ab + (a - 1)(b - 1) \in \mathbb{N}$). Thus, the multiplication of two odd numbers is an odd number. Again, $1 \in F$, and for any $x \in F$, $1 \cdot x = x \cdot 1 = x$. Hence, the identity element exists and is 1.

   (c) Yes. Addition is a binary operation on E. We see that $2a + 2b = 2(a + b)$ for all $a, b \in \mathbb{N}$. Thus, the addition of two even numbers is an even number. Again, if e is the identity element, then for any $x \in E$, we must have $e + x = x + e = x$. There is no such e in E (since $0 \notin E$ and $e + x = x + e > x$ for all $e, x \in E$). Hence, the identity element does not exist.

   (d) Yes. Multiplication is a binary operation on E. We see that $2a \times 2b = 2(2ab)$ for all $a, b \in \mathbb{N}$. Thus, the multiplication of two even numbers is an even number. Again, if e is the identity element, then for any $x \in E$, we must have $e \times x = x \times e = x$. There is no such e in E (since $1 \notin E$ and $e \times x = x \times e > x$ for all $e, x \in E$). Hence, the identity element does not exist.

2. State whether each of the following definitions of $*$ gives a binary operation on $\mathbb{N}$ or not. Give justification of your answer.

   (i) $a * b = a - b$, **(ii)** $a * b = |a - b|$, **(iii)** $a * b = a^2 b$,
   (iv) $a * b = b$, **(v)** $a * b = a + ab$, **(vi)** $a * b = a^b$,
   (vii) $a * b = ab - 1$, **(viii)** $a * b = ab + 1$.

Solution: **(i)** No. We see that $1, 3 \in \mathbb{N}$, but $1 * 3 = 1 - 3 = -2 \notin \mathbb{N}$.

(ii) No. We see that $1, 1 \in \mathbb{N}$, but $1 * 1 = 1 - 1 = 0 \notin \mathbb{N}$.

(iii) Yes. Since multiplication is a binary operation on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $a * b = a^2 b = a \times a \times b \in \mathbb{N}$.

(iv) Yes. For any $a, b \in \mathbb{N}$, $a * b = b \in \mathbb{N}$.

(v) Yes. Since multiplication is a binary operation on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $ab \in \mathbb{N}$. Again, since addition is a binary operation on $\mathbb{N}$, $a * b = a + ab \in \mathbb{N}$.

(vi) Yes. Since multiplication is a binary operation on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $a * b = a^b = \underbrace{a \times a \times \cdots \times a}_{b \text{ times}} \in \mathbb{N}$.

(vii) No. We see that $1, 1 \in \mathbb{N}$, but $1 * 1 = 1 \times 1 - 1 = 0 \notin \mathbb{N}$.

(viii) Yes. Since multiplication and addition are binary operations on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $a * b = ab + 1 \in \mathbb{N}$.

3. Prove that the following binary operations on $\mathbb{N}$ are commutative but not associative.

(i) $a * b = 2a + 2b$,	**(ii)** $a * b = 2^{ab}$,
(iii) $a * b = (a - b)^2$,	**(iv)** $a * b = ab + 1$.

Solution: **(i)** Since addition is commutative on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $a * b = 2a + 2b = 2b + 2a = b * a$. Hence, the binary operation $*$ is commutative on $\mathbb{N}$.

Taking $a = 1$, $b = 1$, $c = 2$, we have

$$a * (b * c) = 1 * (1 * 2) = 1 * (2 \cdot 1 + 2 \cdot 2) = 1 * 6 = 2 \cdot 1 + 2 \cdot 6 = 14,$$
$$(a * b) * c = (1 * 1) * 2 = (2 \cdot 1 + 2 \cdot 1) * 2 = 4 * 2 = 2 \cdot 4 + 2 \cdot 2 = 12.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{N}$. Hence, the operation $*$ is not associative. □

(ii) Since multiplication is commutative on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $ab = ba$ and so $a * b = 2^{ab} = 2^{ba} = b * a$. Hence, the binary operation $*$ is commutative on $\mathbb{N}$.

Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * (2^{2 \times 3}) = 1 * 64 = 2^{1 \times 64} = 2^{64},$$
$$(a * b) * c = (1 * 2) * 3 = (2^{1 \times 2}) * 3 = 4 * 3 = 2^{4 \times 3} = 2^{12}.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{N}$. Hence, the operation $*$ is not associative. □

(iii) When $a, b \in \mathbb{N}$ and $a = b$, we see that $a * b = (a - b)^2 = 0 \notin \mathbb{N}$. Hence, $*$ is not a binary operation on $\mathbb{N}$. (cf. Example 1, page 3.) □

(iv) Since multiplication is commutative on $\mathbb{N}$, for any $a, b \in \mathbb{N}$, $ab = ba$ and so $a * b = ab + 1 = ba + 1 = b * a$. Hence, the binary operation $*$ is commutative on $\mathbb{N}$.

Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 \times 3 + 1) = 1 * 7 = 1 \times 7 + 1 = 8,$$
$$(a * b) * c = (1 * 2) * 3 = (1 \times 2 + 1) * 3 = 3 * 3 = 3 \times 3 + 1 = 10.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{N}$. Hence, the operation $*$ is not associative. □

4. Show that the binary operation $*$ on $\mathbb{N}$ defined by $a * b = b$ is associative but not commutative.

Solution: For any $a, b, c \in \mathbb{N}$, we have

$$a * (b * c) = a * c = c \text{ and } (a * b) * c = b * c = c.$$

Thus, $a * (b * c) = (a * b) * c \; \forall \; a, b, c \in \mathbb{N}$. Hence, the given binary operation $*$ is associative.

Taking $a = 1$ and $b = 2$, we have

$$a * b = 1 * 2 = 2 \text{ and } b * a = 2 * 1 = 1.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{N}$. Hence, the binary operation $*$ is not commutative. □

5. Show that the following binary operations $*$ on $\mathbb{Q}$ are neither associative nor commutative.

(i) $x * y = x - y + 1$, **(ii)** $x * y = 2x + 3y$,
(iii) $x * y = x + xy$, **(iv)** $x * y = xy^2$.

Solution: **(i)** Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 - 3 + 1) = 1 * 0 = 1 - 0 + 1 = 2,$$
$$(a * b) * c = (1 * 2) * 3 = (1 - 2 + 1) * 3 = 0 * 3 = 0 - 3 + 1 = -2.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{Q}$. Hence, the operation $*$ is not associative.

Again, for $a = 1$ and $b = 2$, we have

$$a * b = 1 * 2 = 1 - 2 + 1 = 0 \text{ and } b * a = 2 * 1 = 2 - 1 + 1 = 2.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{Q}$. Hence, the given binary operation $*$ is not commutative. □

(ii) Taking $a = 1$, $b = 0$, $c = 0$, we have

$$a * (b * c) = 1 * (0 * 0) = 1 * (2 \cdot 0 + 3 \cdot 0) = 1 * 0 = 2 \cdot 1 + 3 \cdot 0 = 2,$$
$$(a * b) * c = (1 * 0) * 0 = (2 \cdot 1 + 3 \cdot 0) * 0 = 2 * 0 = 2 \cdot 2 + 3 \cdot 0 = 4.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{Q}$. Hence, the operation $*$ is not associative.

Again, for $a = 1$ and $b = 2$, we have

$$a * b = 1 * 2 = 2 \times 1 + 3 \times 2 = 8 \text{ and } b * a = 2 * 1 = 2 \times 2 + 3 \times 1 = 7.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{Q}$. Hence, the given binary operation $*$ is not commutative. □

(iii) Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 + 2 \times 3) = 1 * 8 = 1 + 1 \times 8 = 9,$$
$$(a * b) * c = (1 * 2) * 3 = (1 + 1 \times 2) * 3 = 3 * 3 = 3 + 3 \times 3 = 12.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{Q}$. Hence, the operation $*$ is not associative.

Again, for $a = 1$ and $b = 2$, we have

$$a * b = 1 * 2 = 1 + 1 \times 2 = 3 \text{ and } b * a = 2 * 1 = 2 + 2 \times 1 = 4.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{Q}$. Hence, the given binary operation $*$ is not commutative. □

(iv) Taking $a = 1$, $b = 2$, $c = 3$, we have

$$a * (b * c) = 1 * (2 * 3) = 1 * (2 \times 3^2) = 1 * 18 = 1 \times 18^2 = 18^2,$$
$$(a * b) * c = (1 * 2) * 3 = (1 \times 2^2) * 3 = 4 * 3 = 4 \times 3^2 = 6^2.$$

Thus, we see that $a * (b * c) \neq (a * b) * c$, for some $a, b, c \in \mathbb{Q}$. Hence, the operation $*$ is not associative.

Again, for $a = 1$ and $b = 2$, we have

$$a * b = 1 * 2 = 1 \times 2^2 = 4 \text{ and } b * a = 2 * 1 = 2 \times 1^2 = 2.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{Q}$. Hence, the given binary operation $*$ is not commutative. □

6. Prove that the binary operation $\circ$ on $\mathbb{Z}$ defined by $a \circ b = a + b - 5$ is associative as well as commutative.

   Solution: For any $x, y, z \in \mathbb{Z}$,

   $$x \circ (y \circ z) = x \circ (y + z - 5) = x + (y + z - 5) - 5 = x + y + z - 10,$$
   $$(x \circ y) \circ z = (x + y - 5) \circ z = (x + y - 5) + z - 5 = x + y + z - 10.$$

   Thus, we see that $x \circ (y \circ z) = (x \circ y) \circ z$ for all $x, y, z \in \mathbb{Z}$. Hence, the binary operation $\circ$ is associative.

   Again, for any $x, y \in \mathbb{Z}$, we have

   $$x \circ y = x + y - 5 = y + x - 5 = y \circ x.$$

   Hence, the binary operation $\circ$ is commutative. □

7. Prove that the binary operation $*$ on $\mathbb{Z}$ defined by $a * b = 3a + 5b$ is neither associative nor commutative. Also, prove that the usual multiplication on $\mathbb{Z}$ distributes over $*$.

   Solution: Here, the given binary operation on $\mathbb{Z}$ is $a * b = 3a + 5b$. Taking $x = 1$, $y = 0$ and $z = 0$, we have

   $$x * (y * z) = 1 * (0 * 0) = 1 * (3 \cdot 0 + 5 \cdot 0) = 1 * 0 = 3 \cdot 1 + 5 \cdot 0 = 3,$$
   $$(x * y) * z = (1 * 0) * 0 = (3 \cdot 1 + 5 \cdot 0) * 0 = 3 * 0 = 3 \cdot 3 + 5 \cdot 0 = 9.$$

   Thus, we see that $x * (y * z) \neq (x * y) * z$ for some $x, y, z \in \mathbb{Z}$. Hence, $*$ is not associative on $\mathbb{Z}$.

   Again, for $x = 0$ and $y = 3$, we have

   $$x * y = 0 * 3 = 3 \times 0 + 5 \times 3 = 15 \text{ and } y * x = 3 * 0 = 3 \times 3 + 5 \times 0 = 9.$$

   Thus, we see that $x * y \neq y * x$ for some $x, y \in \mathbb{Z}$. Hence, $*$ is not commutative.

   Now, for any $x, y, z \in \mathbb{Z}$, we have

   $$x(y * z) = x(3y + 5z) = 3xy + 5xz,$$
   $$(xy) * (xz) = 3xy + 5xz.$$

   Similarly, for any $x, y, z \in \mathbb{Z}$, we have

   $$(x * y)z = (3x + 5y)z = 3xz + 5yz,$$
   $$(xz) * (yz) = 3xz + 5yz.$$

   Thus, $x(y * z) = (xy) * (xz)$ and $(x * y)z = (xz) * (yz)$ for all $x, y, z \in \mathbb{Z}$. Hence, the usual multiplication on $\mathbb{Z}$ distributes over the binary operation $*$ on $\mathbb{Z}$. □

8. Let the binary operations $\circ$ and $*$ on $\mathbb{R}$ be defined by

$$x \circ y = 2x + 2y \text{ and } x * y = x.$$

Show that $\circ$ is commutative but not associative and $*$ is associative but not commutative. Also, show that $\circ$ distributes over $*$.

Solution: For any $a, b \in \mathbb{R}$, we have

$$a \circ b = 2a + 2b = 2b + 2a = b \circ a.$$

Hence, $\circ$ is commutative on $\mathbb{R}$. Now, for $a = 0$, $b = 1$ and $c = 2$, we have

$$a \circ (b \circ c) = 0 \circ (1 \circ 2) = 0 \circ (2 \cdot 1 + 2 \cdot 2) = 0 \circ 6 = 2 \cdot 0 + 2 \cdot 6 = 12,$$
$$(a \circ b) \circ c = (0 \circ 1) \circ 2 = (2 \cdot 0 + 2 \cdot 1) \circ 2 = 2 \circ 2 = 2 \cdot 2 + 2 \cdot 2 = 8.$$

Thus, $a \circ (b \circ c) \neq (a \circ b) \circ c$ for some $a, b, c \in \mathbb{R}$. Therefore, $\circ$ is not associative.

For any $a, b, c \in \mathbb{R}$, we have

$$a * (b * c) = a * b = a,$$
$$(a * b) * c = a * c = a.$$
$$\therefore a * (b * c) = (a * b) * c \ \forall \ a, b, c \in \mathbb{R}.$$

Hence, $*$ is associative. Now, for $a = 0$ and $b = 1$, we have

$$a * b = 0 * 1 = 0 \text{ and } b * a = 1 * 0 = 1.$$

Thus, we see that $a * b \neq b * a$ for some $a, b \in \mathbb{R}$. Therefore, $*$ is not commutative.

Now, for any $a, b, c \in \mathbb{R}$, we have

$$a \circ (b * c) = a \circ b = 2a + 2b = (2a + 2b) * (2a + 2c) = (a \circ b) * (a \circ c).$$

Similarly, for any $a, b, c \in \mathbb{R}$, we have

$$(a * b) \circ c = a \circ c = 2a + 2c = (2a + 2c) * (2b + 2c) = (a \circ c) * (b \circ c).$$

Hence, $\circ$ distributes over $*$. □

9. Prove that the binary operation $\circ$ on $\mathbb{N}$ defined by $a \circ b =$ maximum of a and b is associative and commutative. Find the identity element and the invertible elements of $(\mathbb{N}, \circ)$.

Solution: For any $a, b, c \in \mathbb{N}$, we have

$$\begin{aligned} a \circ (b \circ c) &= a \circ d, \text{ where } d \text{ is the maximum of } b \text{ and } c \\ &= \text{maximum of } a \text{ and } d \end{aligned}$$

$$= \text{maximum of } a,\ b \text{ and } c.$$

Similarly, $(a \circ b) \circ c =$ maximum of a, b and c. Thus, $a \circ (b \circ c) = (a \circ b) \circ c$ for all $a, b, c \in \mathbb{N}$. Hence, $\circ$ is associative.

For any $a, b \in \mathbb{N}$, we have

$$a \circ b = \text{maximum of } a \text{ and } b = \text{maximum of } b \text{ and } a = b \circ a.$$

Hence, $\circ$ is commutative. □

If e is the identity element, then for any $a \in \mathbb{N}$, we have

$$\begin{aligned} e \circ a &= a \circ e = a \\ \implies &\text{maximum of } e \text{ and } a = a \\ \implies &e \leq a\ \forall\ a \in \mathbb{N}. \end{aligned}$$

This inequality holds only when $e = 1 \in \mathbb{N}$. Hence, 1 is the identity element of $(\mathbb{N}, \circ)$. Again, for any $a \in \mathbb{N}$, let b be the inverse of a. Then

$$a \circ b = 1 \implies \text{maximum of } a \text{ and } b = 1.$$

This is possible only when $a = b = 1$. Hence, 1 is the only invertible element of $(\mathbb{N}, \circ)$.

10. Investigate the set of integers, the set of rational numbers and the set of irrational numbers for the closure under the following binary operations: **(i)** addition, **(ii)** subtraction, **(iii)** multiplication, **(iv)** division.

Solution: Consider the set $\mathbb{Z}$ of integers. We know that addition, subtraction or multiplication of two integers is again an integer. So, the set of integers is closed under addition, subtraction and multiplication. Since $1, 2 \in \mathbb{Z}$ and $1/2 \notin \mathbb{Z}$, the set of integers is not closed under division.

Again, consider the set $\mathbb{Q}$ of rational numbers. We know that for any two rational numbers $x = a/b$ and $y = c/d$, where $a, b, c, d \in \mathbb{Z}$, $b \neq 0$, $d \neq 0$, we have

$$\frac{a}{b} \pm \frac{c}{d} = \frac{ad \pm bc}{bd},\ bd \neq 0; \qquad x \times y = \frac{a}{b} \times \frac{c}{d} = \frac{ac}{bd},\ bd \neq 0.$$

Since $\mathbb{Z}$ is closed under multiplication, addition and subtraction, we see that $ad \pm bc, ac, bd \in \mathbb{Z}, bd \neq 0$. So, $x \pm y, xy \in \mathbb{Q}$ for all $x, y \in \mathbb{Q}$. Hence, the set of rational numbers is closed under addition, subtraction and multiplication.

Since $1, 0 \in \mathbb{Q}$ and $1/0 \notin \mathbb{Q}$, the set of rational numbers is not closed under division.

Lastly, consider the set $\mathbb{Q}^c$ of irrational numbers. Let $a = 1 + \sqrt{2}$

and $b = 1 - \sqrt{2}$. Now, $a + b = (1 + \sqrt{2}) + (1 - \sqrt{2}) = 2$, $a - a = 0$, $ab = (1 + \sqrt{2})(1 - \sqrt{2}) = -1$ and $a/a = 1$. Now, we see that $a, b \in \mathbb{Q}^c$, whereas $a + b, a - a, ab, a/a \notin \mathbb{Q}^c$. Hence, the set of irrational numbers is not closed under addition, subtraction, multiplication and division.

11. Prove that there is no non-empty finite subset of $\mathbb{N}$ closed under addition.

    Solution: Let A be any non-empty subset of $\mathbb{N}$ closed under addition. Since A is non-empty, there exists a natural number $a \in A$. Again, A is closed under addition. So, we have

    $$\begin{aligned} a + a &= 2a \in A, \\ 2a + a &= 3a \in A, \\ &\vdots \end{aligned}$$

    Also, $a < 2a < 3a < \cdots$. Thus, the natural numbers $a, 2a, 3a, \ldots$ are distinct elements in A. So, A is infinite. Now, we have shown that any non-empty subset of $\mathbb{N}$ closed under addition is infinite. Hence, there is no non-empty finite subset of $\mathbb{N}$ closed under addition. $\square$

    Or

    Let A be any non-empty finite subset of $\mathbb{N}$. Since we can arrange any given natural numbers in ascending order, we may take

    $$A = \{a_1 < a_2 < \cdots < a_k\}.$$

    But $a_1 + a_k \notin A$ (because $a_1 + a_k > a_k$). Thus, A is not closed under addition. So, any non-empty finite subset of $\mathbb{N}$ is not closed under addition. Hence, there is no non-empty finite subset of $\mathbb{N}$ closed under addition. $\square$

12. Prove that the only non-empty finite subset of $\mathbb{N}$ closed under multiplication is $\{1\}$.

    Solution: Consider the subset $\{1\}$ of $\mathbb{N}$. Since $1 \times 1 = 1 \in \{1\}$, the subset $\{1\}$ is closed under multiplication. Let, if possible, A $(\neq \{1\})$ be any non-empty finite subset of $\mathbb{N}$ closed under multiplication. Since A is non-empty and $A \neq \{1\}$, there exists a natural number $a \in A$ different from 1 (i.e., $a \neq 1$). Again, A is closed under multiplication. So, we have

    $$\begin{aligned} a \times a &= a^2 \in A, \\ a^2 \times a &= a^3 \in A, \end{aligned}$$

⋮

Since $a \neq 1$ and $a \in \mathbb{N}$, $a < a^2 < a^3 < \cdots$. Thus, the natural numbers $a, a^2, a^3, \ldots$ are distinct elements in A. So, A is infinite, which is a contradiction. Therefore, such A does not exist. Hence, the only non-empty finite subset of $\mathbb{N}$ closed under multiplication is $\{1\}$. □

13. Find whether the identity element exists or not for each of the following algebraic structures.

(i) $(\mathbb{N}, +)$, **(ii)** $(\mathbb{N}, \cdot)$, **(iii)** $(\mathbb{Z}, +)$, **(iv)** $(\mathbb{Z}, \cdot)$,
(v) $(\mathbb{Q}, +)$, **(vi)** $(\mathbb{Q}, \cdot)$, **(vii)** $(P(S), \cap)$, **(viii)** $(P(S), \cup)$.

Solution: (i) If e is the identity element of $(\mathbb{N}, +)$, then we must have

$$a + e = e + a = a \text{ for all } a \in \mathbb{N}.$$

In particular, we must have $1 + e = 1$. This is no such $e \in \mathbb{N}$. (Note that $0 \notin \mathbb{N}$). Hence, the identity element does not exist for $(\mathbb{N}, +)$.

(ii) If e is the identity element of $(\mathbb{N}, \cdot)$, then we must have

$$a \cdot e = e \cdot a = a \text{ for all } a \in \mathbb{N}.$$

This is true when $e = 1 \in \mathbb{N}$. Hence, the identity element exists for $(\mathbb{N}, \cdot)$ and is 1.

(iii) If e is the identity element of $(\mathbb{Z}, +)$, then we must have

$$a + e = e + a = a \text{ for all } a \in \mathbb{Z}.$$

This is true when $e = 0 \in \mathbb{Z}$. Hence, the identity element exists for $(\mathbb{Z}, +)$ and is 0.

(iv) If e is the identity element of $(\mathbb{Z}, \cdot)$, then we must have

$$a \cdot e = e \cdot a = a \text{ for all } a \in \mathbb{Z}.$$

This is true when $e = 1 \in \mathbb{Z}$. Hence, the identity element exists for $(\mathbb{Z}, \cdot)$ and is 1.

(v) If e is the identity element of $(\mathbb{Q}, +)$, then we must have

$$a + e = e + a = a \text{ for all } a \in \mathbb{Q}.$$

This is true when $e = 0 \in \mathbb{Q}$. Hence, the identity element exists for $(\mathbb{Q}, +)$ and is 0.

(vi) If e is the identity element of $(\mathbb{Q}, \cdot)$, then we must have

$$a \cdot e = e \cdot a = a \text{ for all } a \in \mathbb{Z}.$$

This is true when $e = 1 \in \mathbb{Q}$. Hence, the identity element exists for $(\mathbb{Q}, \cdot)$ and is 1.

(vii) If E is the identity element of $(P(S), \cap)$, then we must have

$$\begin{aligned} A \cap E = E \cap A = A \text{ for all } A \in P(S) \\ \implies A \subseteq E \text{ for all } A \in P(S). \end{aligned}$$

This implies that $E = S \in P(S)$. Hence, the identity element exists for $(P(S), \cap)$ and is S.

(viii) If E is the identity element of $(P(S), \cup)$, then we must have

$$\begin{aligned} A \cup E = E \cup A = A \text{ for all } A \in P(S) \\ \implies E \subseteq A \text{ for all } A \in P(S). \end{aligned}$$

This implies that $E = \phi \in P(S)$. Hence, the identity element exists for $(P(S), \cup)$ and is ϕ.

14. Let $S = \{1, 2, 3, 4, 5, 6, 7\}$. Find the identity element of the algebraic structure $(P(S), \cap)$. Also, find the inverse of $A = \{2, 3, 4\}$, if it exists.

Solution: If E is the identity element of $(P(S), \cap)$, then we must have

$$\begin{aligned} A \cap E = E \cap A = A \text{ for all } A \in P(S) \\ \implies A \subseteq E \text{ for all } A \in P(S). \end{aligned}$$

This implies that $E = S \in P(S)$. Hence, the identity element of the algebraic structure $(P(S), \cap)$ is S.

For any element $B \in P(S)$, we have

$$A \cap B = B \cap A \subseteq A \subsetneq S.$$

Thus, $A \cap B = B \cap A \neq S$ for all $B \in P(S)$. Hence, the inverse of A does not exist.

15. Consider the binary operation $*$ on $\mathbb{Q}$ defined by

$$x * y = x + y - xy.$$

Find the identity element of $(\mathbb{Q}, *)$. Also, find x^{-1} for $x \in \mathbb{Q}$. For what value of x does the inverse not exist?

Solution: Let e be the identity element of $(\mathbb{Q}, *)$. Then, for any $a \in \mathbb{Q}$, we must have

$$\begin{aligned} a * e = e * a = a \\ \implies a + e - ae = a \\ \implies e(1 - a) = 0 \end{aligned}$$

$$\implies e = 0, \text{ provided } a \neq 1.$$

So, $a * 0 = 0 * a = a$ for all $a \in \mathbb{Q}$, $a \neq 1$. For $a = 1$, we have

$$a * 0 = 1 * 0 = 1 + 0 - 1 \times 0 = 1$$
$$0 * a = 0 * 1 = 0 + 1 - 0 \times 1 = 1.$$

Thus, we see that $a * 0 = 0 * a = a$ for all $a \in \mathbb{Q}$. Hence, 0 is the identity element of $(\mathbb{Q}, *)$.

Let y be the inverse of $x \in \mathbb{Q}$. Then we must have

$$\begin{aligned} & x * y = y * x = 0 \\ \implies & x + y - xy = 0 \\ \implies & x = y(x-1) \\ \implies & y = \frac{x}{x-1}, \text{ which is defined in } \mathbb{Q} \text{ for all } x \neq 1. \end{aligned}$$

Hence, the inverse of any element $x \in \mathbb{Q}$, other than 1, is $\frac{x}{x-1}$. The inverse of the element 1 does not exist.

16. Form the composition table for the set $S = \{1, 2, 3, 4, 5, 6\}$ with respect to the binary operation of multiplication modulo 7. Deduce that S is closed under the operation. From the table, find the identity element and the inverse of each element of S. Also, calculate 2^6 in S.

Solution: We form the composition table for $(S, \times_7)$ as follows.

$\times_7$	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

From the table, we see that $x \times_7 y \in S$ for all $x, y \in S$. So, S is closed under the binary operation $\times_7$. We also see from the table that 1 is the identity element of S and $1^{-1} = 1$, $2^{-1} = 4$, $3^{-1} = 5$, $4^{-1} = 2$, $5^{-1} = 3$, $6^{-1} = 6$. Now, $2^6 = (2^2)^3 = (2 \times_7 2)^3 = 4^3 = (4 \times_7 4) \times_7 4 = 2 \times_7 4 = 1$.

17. Form the composition table for the set $S = \{0, 1, 2, 3, 4, 5\}$ with respect to the binary operation of addition modulo 6. From the table, find the identity element and the inverse of each element of S.

Solution: We form the composition table for $(S, +_6)$ as follows.

$+_6$	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	0
2	2	3	4	5	0	1
3	3	4	5	0	1	2
4	4	5	0	1	2	3
5	5	0	1	2	3	4

From the table, we see that $0 +_6 a = a$ and $a +_6 0 = a$ for all $a \in S$. Hence, 0 is the identity element of S.

The inverse of an element $a \in S$ is the element b of S, where $a +_6 b = 0$ and $b +_6 a = 0$. From the table, we see that $0^{-1} = 0$, $1^{-1} = 5$, $2^{-1} = 4$, $3^{-1} = 3$, $4^{-1} = 2$, $5^{-1} = 1$.

Remark: In order to find the inverse of an element $a \in S$, look at the row headed by a, and find out the the entry and the respective column where the identity element 0 is. Then the element that headed this particular column is the inverse of a.

18. Let a binary operation $*$ on $\mathbb{N}$ be defined by

$$a * b = \text{HCF of } a \text{ and } b.$$

Show by means of a composition table that the set $H = \{1, 2, 3, 4, 5, 6\}$ is closed under $*$.

Solution: We form the composition table for $(H, *)$ as follows.

$*$	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

From the table, we see that $x * y \in H$ for all $x, y \in H$. Hence, H is closed under the binary operation $*$.

19. A binary operation $\circ$ on $\mathbb{N}$ is defined by

$$a \circ b = \text{LCM of } a \text{ and } b.$$

Form the composition table for the set $H = \{1, 2, 3, 4, 5\}$ with respect to $\circ$. State whether H is closed under $\circ$ or not.

Solution: We form the composition table for $(H, \circ)$ as follows.

$\circ$	1	2	3	4	5
1	1	2	3	4	5
2	2	2	6	4	10
3	3	6	3	12	15
4	4	4	12	4	20
5	5	10	15	20	5

From the table, we see that $2 \circ 3 = 6 \notin H$. Hence, H is not closed under the binary operation $\circ$.

20. Prove that the set $S = \{3n : n \in \mathbb{Z}\}$ is closed under the usual addition and multiplication. Examine the algebraic structures $(S, +)$ and $(S, \cdot)$ for existence of identity and invertible elements.

Solution: Let x and y be any two elements of S. Then $x = 3m$ and $y = 3n$ for some $m, n \in \mathbb{Z}$. Now, we have

$$x + y = 3m + 3n = 3(m + n) \in S \text{ and } x \cdot y = 3m \cdot 3n = 3(3mn) \in S.$$

Thus, $x + y$, $x \cdot y \in S$ for all $x, y \in S$. Hence, S is closed under the usual addition and multiplication. □

Consider the algebraic structure $(S, +)$. Let e be the identity element of S. Then for any element $a = 3m$ of S, we have

$$a + e = e + a = a \implies 3m + e = 3m \implies e = 0 = 3 \times 0 \in S.$$

Hence, 0 is the identity element of $(S, +)$. Again, if y is the inverse of an element $x = 3m$ of S, then

$$x + y = y + x = 0 \implies 3m + y = 0 \implies y = -3m = 3(-m) \in S.$$

Hence, the inverse of any element $3m$ is $-3m$.

Consider the algebraic structure $(S, \cdot)$. Let e be the identity element of S. Then for any element $a = 3m$ of S, we have

$$a \cdot e = e \cdot a = a \implies 3me = 3m.$$

In particular, when $m = 1$, we get $3e = 3$. This is not possible for any element $e \in S$. Hence, the identity element does not exist for the algebraic structure $(S, \cdot)$ and as a result, the inverse does not exist for any element.

Chapter 8

Statics

> *"This most beautiful system of the sun, planets, and comets, could only proceed from the counsel and dominion of an intelligent and powerful Being."*
>
> — Isaac Newton, *The Principia: Mathematical Principles of Natural Philosophy*

Mechanics is the branch of science which deals with the action of forces on bodies. In this chapter, we shall study the basics of statics. Statics is the branch of mechanics which deals with bodies at rest and forces in equilibrium.

Some Terms and Definitions

Definition 8.1 (Matter). A matter is anything that occupies space and can be perceived by our senses.

Definition 8.2 (Body). A body is a portion of a matter limited in all directions, having a definite shape and size, and occupying some definite space.

Definition 8.3 (Force). A force is that which changes or tends to change the state of rest or of uniform motion of a body.

Definition 8.4 (Rigid body). A rigid body is one whose size and shape do not alter when acted on by any forces whatsoever, so that the distance between any pair of particles in it remains invariable.

Definition 8.5 (Particle). A particle is a body of infinitely small dimensions. When we speak of a body as a particle, we mean that we are not concerned with its actual dimensions and that its position can be represented by a mathematical point.

Definition 8.6 (Equilibrium). If a system of forces acting on a body keeps it at rest, then the forces are said to be in equilibrium.

Representation of a Force

A force is generally characterized by its point of application, its magnitude, and its direction. It is a vector quantity. It can be represented by a line segment ending with an arrowhead. The length of the line segment is proportional to the magnitude of the force. The *magnitude* refers to the size or amount of the force in acceptable units. The SI unit of force is newton (N).

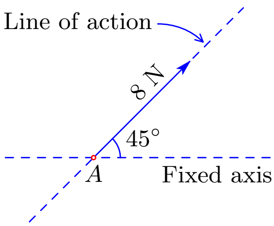


Figure 1

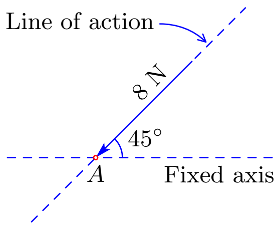
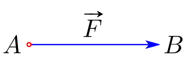


Figure 2

The *line of action* of a force is the path of the line along which the force acts. The *sense* of a force refers to the way in which the force acts along its line of action. The sense is represented by an arrowhead. A force may be acting vertically, but the sense could be up or down. Figure 1 and Figure 2 show two forces having the same magnitude (8 N), the same point of application (point A), and the same line of action (45° from the horizontal) but a different sense. The line of action and the sense determine the *direction* of the force. The word 'direction' is sometimes used to refer to the line of action of the force.



A force represented by a line segment AB with sense from A to B is denoted in vector notation by $\overrightarrow{AB}$. If the sense is from B to A, then the force is denoted by $\overrightarrow{BA}$. The magnitude of a force $\vec{F}$ is denoted by F. It should be noted that the magnitude of a force is always non-negative. In solving problems, the sense of an unknown force, say $\vec{P}$, along its line of action may be chosen arbitrarily. In such cases, the value of P thus calculated represents the magnitude with the sense; a negative sign indicates that the sense of $\vec{P}$ is opposite the sense we have chosen.

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